Classical Electrodynamics

(Lecture Notes, Perpetually in progress)

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Overview

This course introduces the classical theory of electrodynamics describing the interactions of charged particles among themselves and with electromagnetic fields. Understanding these interactions is necessary for studies of elementary particles, charged fluids, electronics used in many areas such as communication and in general technological devices. The formulation of the theory in terms of Maxwell's equations is the starting point of many research avenues in modern Physics and Mathematics.

Contents

1 Historical remarks and motivation

The etymology of the word Electrodynamics comes from the Greek words "$\epsilon\lambda\epsilon\kappa\tau\rho o$" (electro, an alloy of gold and silver) and $\delta\upsilon\nu\alpha\mu\iota\kappa\eta$ (dynamics). Electrodynamics is an area of Physics dealing, generally speaking, with the interactions between electric charges. All technological developments we are accustomed to, are nothing but applications, of that theory (most of the time in conjunction with Quantum Mechanics). This means nothing but clever ways to use these interactions to our benefit, i.e. make our lives easier, or at least different.

A synonym to Electrodynamics is Electromagnetism, a synthesis of Electricity and Magnetism dealing with electric and magnetic phenomena, which are, as it will be explained, nothing but two facets of the same theory. The development of the theory is ultimate connected with names such as the French Physicist Charles-Augustin de Coulomb (1736-1806) and the British Physicists Michael Faraday(1791-1867) and James Clerk Maxwell (1831-1879). The final formulation of the theory in terms of the Maxwell's equations is aesthetically one of the most beautiful set of partial differential equations ever discovered. Moreover, this system helped Einstein to formulate his Special Theory of Relativity. In addition, the concept of gauge invariance which all modern theories of elementary particles use as an essential incredient has its roots in Classical Electrodynamics.

The word classical in the name of this module emphasizes essentially the validity of the theory that we will develop and analyze. Roughly speaking it is valid for distances larger than $10^{-9}m$. At smaller distances quantum effects become important. The quantum extension of Classical Electrodynamics is called Quantum Electrodynamics(QED) and is considered one of the cornerstones of scientific achievements. This will not be the subject of the present module.

The theory of Electrodynamics is the starting point of many research avenues in modern Physics and Mathematics. By that I mean that one can use its tools and concepts in different branches of Mathematical and Physical Sciences from the more theoretical to direct applications.

At the end of the module a student should have a firm understanding:

• of the notion of the electric and magnetic fields as well as of the associated scalar and vector potentials,

• of the nature of forces charged particles and bodies are experiencing in electric and magnetic fields,

• how unification of electric and magnetic phenomena is achieved via the unifying description by the Maxwell equations,

• how to solve boundary value problems in static cases and

• of the fields of moving charges and how radiation phenomena arise.

Abbreviations-Notation

• l.h.s. (r.h.s.): left (right) hand side.

• w.r.t.: with respect to.

• i.e.: "in a sense" or "in other words".

• e.g.: "for example“.

• b.c.: boundary condition(s).

• Bold face latters, e.g. **A**, will denote vectors. The magnitude of **A** will be denoted by A or by $|\mathbf{A}|$.

• Summation convention: Repeated indices, unless otherwise indicated, are summed. For example, $\sum_{i=1}^{N} A_i B_i$ will be denoted by $A_i B_i$. This is called Einstein's summation convention.

• We will denote the partial derivative w.r.t. x by $\frac{\partial}{\partial x}$, or by ∂_x, or by $\partial/\partial x$ and that w.r.t. x_i, where i some discrete enumeration index, by ∂_i.

• $f(x) = f(0) + \mathcal{O}(x^n)$ means that, as $x \to 0$, the first correction to the leading order term $f(0)$ in a Taylor expansion, is the term x^n (up to a constant), where $n > 0$.

• The symbol c will denote the speed of light, which in vacuum is $c = 3 \times 10^8 \frac{m}{\text{sec}}$.

2 Electrostatics

2.1 Coulomb's law

Some particles when they are brought close enough interact by exerting forces on each other due to a common property they share which is called *charge* and is characterized by a real number.

For two such *static point particles* at a distance r apart, with charges $q_i, i = 1, 2$ the force between them has magnitude inversely proportional to the square of their distance and points along the line connecting them. Mathematically, the force that the particle 1 exerts on particle 2 is

$$\mathbf{F}_{12,e} = \frac{q_1 q_2}{r^2}\hat{r} \, , \tag{2.1}$$

where $\hat{r}$ is the unit vector from 1 to 2 pointing outwards. The force is repulsive (attractive) if the charges have the same (opposite) signs. This is *Coulomb's law*. The force that particle 2 exerts on particle 1 is $\mathbf{F}_{21,e} = -\mathbf{F}_{12,e}$, in agreement with the action-reaction principle of classical mechanics.

Coulomb's law has been experimentally verified to a very high accuracy. For instance, by replacing the radial dependence in (2.1) by (i) $\frac{1}{r^{2+\epsilon}}$ and (ii) $\frac{1}{r}e^{-r/r_0}$ and fitting the pure number parameter ϵ and the length parameter r_0 to experimental results. It has been found that $\epsilon < 10^{-16}$ and $r_0 > 10^8\ m$.

The validity of Coulomb's law breaks down at small distances of the order of $10^{-9}m$ where quantum effects start becoming significant.

2.1.1 System of units and strength of electric force

In principle the r.h.s. of (2.1) could be multiplied by a constant depending on the unit system we are using. Our choice to take it to unity, implies that we are implicitly using the so called *Gaussian* unit system

Physical quantity :	Length	Mass	Time	Force	Energy	Charge
Unit :	*cm*	*gr*	*sec*	*dynes*	*ergs*	*statcoulombs*(sC)

In this unit system the mass of the electron is $m_e = 9.1 \times 10^{-28}\ gr$ and its charge is $q_e = 4.8 \times 10^{-10} sC$.

It is interesting to compare the Coulomb force to the Newton's force of gravitational attraction which has the similar mathematical form

$$\mathbf{F}_{12,gr} = -G_N \frac{m_1 m_2}{r^2} \hat{r} , \tag{2.2}$$

where $G_N = 6.67 \times 10^{-8} \frac{dyn \cdot cm^2}{gr^2}$ is Newton's constant. Comparing the strengths of the two forces

$$\frac{F_{12,gr}}{F_{12,e}} = \frac{G_N m_e^2}{q_e^2} \simeq 1.86 \times 10^{28} , \tag{2.3}$$

demonstrates that the electric force between two electrons is 28 orders of magnitude stronger than the gravitational force. Even more spectacularly, note that the weight of an electron, i.e. the force by which the entire Earth pulls it towards its center is $m_e g \simeq 9 \times 10^{-25} ergs$. The electric force between two electrons $1mm$ apart is $F_{ee} = 2.3 \times 10^{-17} ergs$. Hence the electric force between two electrons even if there are $1mm$ apart is 8 orders of magnitude stronger in strength than the force by which the entire Earth pulls them!

2.1.2 Remarks on charges

• We cannot isolate any smaller amount of charge than the charge of a proton (positive) or electron (negative). We say then that charge is quantized, in the sense that it always comes in discrete amounts, i.e. integer multiples of the charge of the proton or of the electron.

• Charge is conserved. We can move it around, but we can not create nor destroy it.

• All matter is made of charged particles. Typically the positive and negative charges are present in equal numbers. When we say something is charged we mean it possesses a slight imbalance in the number of positive and negative charges.

The *Electric field* due to a static point charge q is defined as

$$\mathbf{E} = \frac{q}{r^2} \hat{r} . \tag{2.4}$$

It equals the force that this charge exerts on a unit charge at distance r away from it.

A natural question is what is the electric field due to several static charges q_i located at points in space with coordinates $\mathbf{r}_i$, where $i = 1, 2, \ldots, N$. Assuming that *linear* *superposition* is possible then we have that

$$\mathbf{E} = \sum_{i=1}^{N} \mathbf{E}_i = \sum_{i=1}^{N} q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} . \tag{2.5}$$

If the distribution of charges is continuous and characterized by a *density* function $\rho(\mathbf{r})$ we have the expression

$$\mathbf{E} = \int_{\mathbb{R}^3} d^3\mathbf{r}'\, \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} . \tag{2.6}$$

From this one can go back to (2.5) by choosing $\rho(\mathbf{r}) = \sum_{i=1}^{N} q_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i)$, where we used the delta-function to describe point changes. Its definition and properties are given in Appendix E.

2.2 Equations for electrostatics

We would like to discover the differential equations obeyed by $\mathbf{E}$ so that we may deal with more complicated situations. We first address the question what kind of differential equations we are after. Assume that we know the divergence and the curl of a vector $\mathbf{V}$

$$\nabla \cdot \mathbf{V} = D\,, \qquad \nabla \times \mathbf{V} = \mathbf{C}\,. \tag{2.7}$$

It looks as if in total D and $\mathbf{C}$ need four independent functions to be determined, but this is not true since $\nabla \cdot \mathbf{C} = 0$ for consistency. Hence, there are three independent functions, precisely as many as the number of components of $\mathbf{V}$. This observation is at the roots of the *Helmholtz theorem* which essentially states that given the divergence and the curl of a vector one can determine the vector itself. There is an extra important ingredient missing, namely the uniqueness of the solution. For instance, both $\mathbf{V} = \mathbf{0}$ and $\mathbf{V} = xy\hat{z} + yz\hat{x} + zx\hat{y}$ have zero divergence and curl, but obviously they differ. According to Helmholtz's theorem a unique solution is obtained provided one specifies for D, $\mathbf{C}$ and $\mathbf{V}$, appropriate falling off b.c. at infinity, i.e. far away from charges.

According to the above, we will proceed to obtain expressions for the divergence and the curl of $\mathbf{E}$ as this is given by (2.6). First note that

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} . \tag{2.8}$$

Then

$$\boxed{\mathbf{E} = -\nabla\Phi}\ , \tag{2.9}$$

where

$$\Phi(\mathbf{r}) = \int_{\mathbb{R}^3} d^3\mathbf{r}'\, \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ . \tag{2.10}$$

For a point charged particle for which $\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r})$ the potential is

$$\Phi(\mathbf{r}) = \frac{q}{r}\ . \tag{2.11}$$

Returning to the general case, from (2.9) we immediately find that

$$\boxed{\nabla\times\mathbf{E} = 0}\ . \tag{2.12}$$

In addition we compute that

$$\nabla\cdot\mathbf{E} = -\nabla^2\Phi = -\int_{\mathbb{R}^3} d^3\mathbf{r}'\rho(\mathbf{r}')\ \underbrace{\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{-4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}')}\ , \tag{2.13}$$

where we have used (E.19). Hence, we find that

$$\boxed{\nabla\cdot\mathbf{E} = 4\pi\rho}\ . \tag{2.14}$$

This is the *differential form* of *Gauss's law* which we will encounter below.

The function Φ is called the *scalar potential*. Substituting (2.9) we find that the scalar potential Φ obeys the *Poisson equation*

$$\boxed{\nabla^2\Phi = -4\pi\rho}\ . \tag{2.15}$$

Note that the fundamental quantity to compute is the electric field $\mathbf{E}$. The potential Φ is not unique, in the sense that if it is replaced by $\Phi + \Psi$, with Ψ obeying the Laplace equation, i.e. $\nabla^2\Psi = 0$, then the electric field remains *invariant*, i.e. unchanged.

2.3 Gauss' law

Consider a volume V in $\mathbb{R}^3$ bounded by a closed surface S. Let the total charge inside this volume be Q which in terms of the density function is given by

$$Q = \int_V d^3\mathbf{r}\, \rho(\mathbf{r}) \ . \tag{2.16}$$

Integrating both sides of (2.14) in that volume and using the divergence theorem we find that

$$\boxed{\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = 4\pi Q} \ , \tag{2.17}$$

which is *Gauss's law* of electrostatics. Sometimes, in parallel with the differential form (2.14), we call it the *integral form of Gauss's law*. Also, note that the l.h.s. of (2.17) defines the *flux of the electric field* through the closed surface S.

Consider any surface S whose boundary is a closed curve C. Then from (2.12) and Stokes's theorem we find that

$$\boxed{\int_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} dS = \oint_C \mathbf{E} \cdot d\mathbf{r} = 0} \ , \tag{2.18}$$

which is consistent with (2.9).

Gauss's law can be used to find the electric field in cases of high symmetry as it is demonstrated in the examples below.

Example: Consider a spherical shell of radius R with total charge Q uniformly distributed over its surface. Due to symmetry the electric field has only radial component in spherical coordinates. Moreover inside the shell there is no charge. Hence, applying Gauss's law on any concentric sphere with radius $r < R$ we conclude that the electric field in the interior of the shell should be zero. A similar sphere but with $r \geqslant R$ encloses all charge, hence

$$\mathbf{E} = \begin{cases} 0\,, & r < R\,, \\ Q/r^2\, \hat{r}\,, & r \geqslant R\,. \end{cases} \tag{2.19}$$

The electric field is discontinuous at $r = R$. As we shall see below in general, the reason for that is related to the discontinuity of the charge density, i.e. charge is only present on the surface of the spherical shell. The potential corresponding to the above electric field is easily found by simply integrating (2.9)

$$\Phi = \begin{cases} Q/R\,, & r \leqslant R\,, \\ Q/r\,, & r \geqslant R\,, \end{cases} \tag{2.20}$$

where we determined the arbitrary constant of integration such that it vanishes for $r \to \infty$ and is continuous for $r = R$.

Example: Consider a sphere of radius R with total charge Q uniformly distributed over its entire volume. As before, due to symmetry the electric field has only a radial component in spherical coordinates, i.e. $\mathbf{E} = E(r)\hat{r}$, where $E(r)$ is a function to be computed. Let's apply Gauss's theorem in a sphere with radius $r \geqslant R$ concentric to the charged sphere. We easily find that $4\pi E(r) r^2 = 4\pi Q$. If instead, the sphere has radius $r \leqslant R$ we find that $4\pi E(r) r^2 = 4\pi \frac{r^3}{R^3} Q$. Hence

$$\mathbf{E} = \begin{cases} Qr/R^3\,\hat{r}\,, & r \leqslant R\,, \\ Q/r^2\,\hat{r}\,, & r \geqslant R\,. \end{cases} \tag{2.21}$$

Note that, unlike (2.19) the electric field is continuous at $r = R$, even though there is a discontinuity in the volume charge density. The corresponding potential is (**exercise**)

$$\Phi = \begin{cases} -\frac{1}{2} Qr^2/R^3 + \frac{3}{2} Q/R\,, & r \leqslant R\,, \\ Q/r\,, & r \geqslant R\,, \end{cases} \tag{2.22}$$

where we have chosen it to vanish for $r \to \infty$.

Exercise: Compute the electric field of a charged sphere of radius R with total charge Q if the charge follows a radially symmetric distribution $\rho(r)$ for $r \leqslant R$.

Exercise: Compute the electric field in the entire space due to a cylinder of length L and radius R such that $L \gg R$, with total charge Q distributed over its entire volume with a density $\rho(\rho)$, for $\rho \leqslant R$. Use this general result to show that for the case of a uniform distribution over the volume of the cylinder the result is

$$\mathbf{E} = \begin{cases} 2\lambda\rho/R^2\,\hat{\rho}\,, & \rho \leqslant R \\ 2\lambda/\rho\,\hat{\rho}\,, & \rho \geqslant R\,, \end{cases} \tag{2.23}$$

where $\lambda = Q/L$ is the charge per unit length. The potential is (**exercise**)

$$\Phi = \begin{cases} \lambda(1-\rho^2/R^2) + C\,, & \rho \leqslant R\,, \\ -2\lambda \ln(\rho/R) + C\,, & r \geqslant R\,, \end{cases} \tag{2.24}$$

In this case we cannot set the potential to zero for $\rho \to \infty$ since the potential does not go to a constant. This is a feature of the solutions of the Laplace eq. in two-dimensions as well as in one dimension. We may arbitrarily however set the constant $C = 0$ which in turn says that the surface of the cylinder is kept at zero potential.

Example: Consider a rectangular sheet with very large sides and constant area charge density σ. Due to symmetry the electric field is of the form $\mathbf{E} = E(z)\hat{z}$, with $E(-z) = -E(z)$. Then apply Gauss's law using a cylinder as in the Fig. 1.

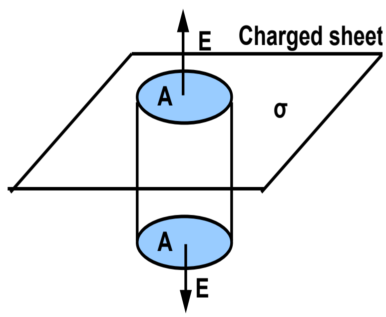

Figure 1: The cylinder has section area A and height $2z$; z is measured from the plane.

We find that $2AE = 4\pi\sigma A$. Hence,

$$\mathbf{E} = 2\pi\sigma \begin{cases} +\hat{z}\,, & z > 0 \\ -\hat{z}\,, & z < 0\,. \end{cases} \tag{2.25}$$

Note again the discontinuity of the electric field at $z = 0$. The potential is simply

$$\Phi = 2\pi\sigma \begin{cases} -z + C\,, & z \geqslant 0 \\ .+z + C\,, & z \leqslant 0\,. \end{cases} \tag{2.26}$$

where C is an arbitrary constant and is of course a continuous function for $z = 0$.

Exercise: Consider a charge Q placed at a vertex of a cube. What is the flux of the electric field through the cube? What about if the charge is at one of its faces or at one of its edges. **Hint:** Try to use symmetry arguments. **Answer:** $\pi Q/2$, $2\pi Q$ and πQ.

Exercise: Consider a charge Q placed at the surface of a sphere. Prove that the flux of the electric field through the surface of the sphere is $2\pi Q$. Use either simple symmetry arguments or the definition of the electric flux.

2.3.1 Gauss's law in two-dimensions

In many instances the electric field has components on the $x - y$ plane only with no z-dependence, i.e. $E_x = E_x(x, y)$, $E_y = E_y(x, y)$ and $E_z = 0$. In that case it makes sense to express the Gauss's law in an appropriate way. Consider as a surface a cylinder with height L and define by $\lambda = Q/L$ the linear charge per unit length. Then since $dS = dcdz$, with dc the displacement along the curve on the boundary of the cylinder, Gauss's law (2.17) becomes

$$\boxed{\oint_C \mathbf{E} \cdot \hat{\mathbf{n}} dc = 4\pi\lambda} \,, \tag{2.27}$$

<u>Exercise</u>: According to (2.23) the electric field of a very long wire of length L with constant linear charge density λ, is given by

$$\mathbf{E} = 2\lambda \frac{\hat{\rho}}{\rho} \,, \qquad \rho \ll L \,. \tag{2.28}$$

Explicitly compute the flux of the electric field through a square due to a charge q placed a) on one of the edges of the square and b) on one of the vertices. How can you find the same result by symmetry arguments?

<u>Exercise</u>: Compute the electric field of an charged wire with length L and constant linear charged density λ at a distance ρ from its midpoint. Show how you recover the field of point charge in the limit $L \ll \rho$ and that in (2.28) in the limit $L \gg \rho$.

<u>Exercise</u>: Consider a charged infinite line with constant density λ placed parallel to an infinite cylinder just touching its surface. Prove either by an explicit computation of by symmetry arguments that the flux of the electric field through the cylinder's surface is $2\pi\lambda$.

2.3.2 Equipotential surfaces and electric lines

Setting the function

$$\Phi(\mathbf{r}) = \text{const.} \,, \tag{2.29}$$

defines a two-dimensional surface which is called *equipotential* surface or *equipotential lines*. This concept plays a very important role as we shall see. If a charged particle

Figure 2: Electric lines of some simple electric fields.

moves along an equipotential surface, then the particle is not gaining or losing any potential energy so that there is no work done on it. Hence the force acting on the particle should be perpendicular to the equipotential surface. Mathematically this is proven by taking the differential of both sides of (2.29) and using (2.9) to get $\mathbf{E}\cdot d\mathbf{r} = 0$, where the displacement is along an equipotential line. This leads to the definition of *electric field lines* as the lines giving the direction of force on a positive charge.

Remarks:

• The electric field lines never start or stop in empty space but either on a charge or at infinity.

• They never cross, since if they did, a small positive charge placed there would feel forces in different directions, which could be resolved into the one true direction of the field line there.

• The density of the field lines on a diagram is indicative of the strength of the electric field. Of course this is only a graphical device.

The electric lines for some simple electric fields are shown in Fig. 2.

2.4 Force, work and energy

When a *test* or *probe charge* q is in an area of space where there is an electric field $\mathbf{E}(\mathbf{r})$ produced by other stationary charges it experiences a force equal to

$$\boxed{\mathbf{F} = q\mathbf{E}}\ . \tag{2.30}$$

For a distribution of charges $\rho(\mathbf{r})$ in a volume V the total force is

$$\boxed{\mathbf{F} = \int_V dv \rho(\mathbf{r})\mathbf{E}(\mathbf{r})}\ . \tag{2.31}$$

We would like to compute the work that we have to do in order to bring this charge from point **a** to the point **b**. The minimum force that we have to use is opposite to that in (2.30). Hence the work needed is

$$W = -q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{r} = q(\Phi(\mathbf{b}) - \Phi(\mathbf{a}))\ . \tag{2.32}$$

Hence, the potential difference between two points equals the work per unit charge required to carry the particle from one point to another.

Assuming that the electric field is localized, i.e. it has appreciable values at a finite region in space and dies off at spatial infinity, the work needed to bring the charge from infinity to the point **r** is

$$W = q\Phi(\mathbf{r})\ , \tag{2.33}$$

where we have set the potential at infinity $\Phi(\infty) = 0$.

Now we address the question how much work is needed to assemble together N point-like charges q_i at fixed positions $\mathbf{r}_i$. Assume that there are already j charges in their positions. Then, the work necessary to bring the next charge in its postion is found by applying (2.33)

$$W_j = q_{j+1}\sum_{i=1}^{j}\frac{q_i}{r_{ij}}\ , \qquad r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|\ . \tag{2.34}$$

Summing over all charges from $j = 1$ to $N-1$ we obtain, after an index rearrangement (**exercise**)

$$W = \sum_{\substack{i,j=1\\ i<j}}^{N}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1} q_i \sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{r_{ij}}\ . \tag{2.35}$$

This is identified with the energy of the configuration of charges.

We would like to give an alterative expression. Interpreting the second sum in the last expression for W above as the potential at $\mathbf{r}_i$ we may write

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{N} q_i\Phi(\mathbf{r}_i)}\ . \tag{2.36}$$

This expression is independent of the way the potential is created and admits a generalization, as we shall see, in cases of continuous distributions of charges.

2.5 Dealing with boundaries and materials

We would like to solve the Poisson equation (2.15) not in free space but in the presence of boundaries that is surfaces where we have to specify the value of the potential Φ or of the electric field $\mathbf{E}$.

2.5.1 Electric field components across a charged surface

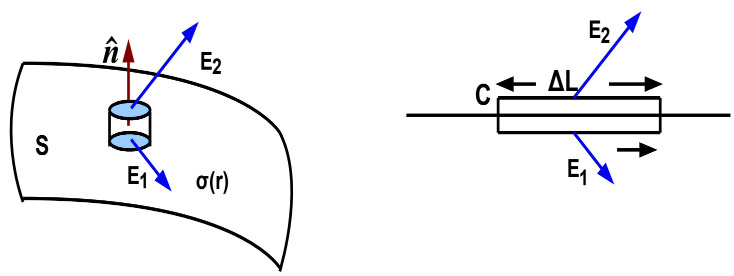

Figure 3: a) The discontinuity of the normal component of the electric field across a surface with charge density $\sigma(\mathbf{r})$. The normal to the surface points from side 1 to 2. b) The continuity of the tangential component of the electric field across a surface.

To find the appropriate b.c. for the electric field on a charged surface, it is convenient to consider a local coordinate system where the normal to the surface $\hat{z}$ is identified with $\hat{\mathbf{n}}$. Then $\nabla \cdot \mathbf{E} = \frac{\partial E_z}{\partial z} + \cdots$. Then consider a small cylinder of volume V surrounding this point with height 2ϵ and section area A. Since the radius is small the electric field is the same in the part of the cylinder above (below) the surface and will be denoted by E_{z_2} ($E_{z,1}$). Then

$$\begin{aligned} &\int_V dv \nabla \cdot \mathbf{E} = A \int_{-\epsilon}^{\epsilon} dz \frac{\partial E_z}{\partial z} + \mathcal{O}(\epsilon) = A(E_{z,2} - E_{z,1}) + \mathcal{O}(\epsilon) \, , \\ &4\pi Q = 4\pi\sigma A + \mathcal{O}(\epsilon) \, . \end{aligned} \tag{2.37}$$

Note that the round side of the cylinder contributes $\mathcal{O}(\epsilon)$ for small ϵ. Equating the l.h.s. according to Gauss's law and in the limit $\epsilon \to 0$ we have that $E_{z,2} - E_{z,1} = 4\pi\sigma$. Passing to a coordinate independent notation we write that

$$\boxed{(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}} = 4\pi\sigma} \, . \tag{2.38}$$

Hence, a surface charge density causes a discontinuity of the electric field. Conversely, a discontinuity of the electric field when crossing a surface, necessarily induces a charge density on it. Note also that (2.25) is consistent with (2.38) and so is (2.19) for uniform surface charge density $\sigma = \dfrac{Q}{4\pi R^2}$.

Unlike the normal component of the electric field, the tangential component is continuous. This is found by applying (2.18) for the path indicated in Fig. 3. We take ΔL small so that the electric field does not change significantly in the parts of the path parallel to the surface. In addition we take the height of the rectangular path to zero. Setting up a local coordinate system so that the horizontal is the x-direction, we obtain that the l.h.s. of (2.18) is $\Delta L(E_{2x} - E_{1x}) = 0$, hence $E_{2x} = E_{1x}$. This condition can be expressed in a coordinate system independent way as

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{E_2} - \mathbf{E_1}) = 0}\ . \tag{2.39}$$

Finally, note that the potential Φ across a surface with a charge density is continuous, as a discontinuity would give rise to an electric field with a delta-function profile (see Appendix E).

2.5.2 Insulators and conductors

In materials the carriers of electric charge are the electrons. When these are mainly localized around particular positions, i.e. in molecules, then the material is called an *insulator*. In the opposite case, when most electrons are movable the material is called a *conductor*. A conductor is called *perfect* or *ideal* if all charge carriers are free to move.

A perfect conductor has several interesting properties admitting a clear mathematical description.

<u>The charge density $\rho = 0$ inside the conductor:</u> Consider a region in a material which is infinitesimal from a mathematical view point but large enough compared to the molecular scale, i.e. larger than $10^{-8}m$. Since charges of equal sign repel each other there cannot be any *net charge* around. There are still charges of opposite signs but of equal magnitudes so that in this region they average to zero.

<u>Only the surface can have charge density $\sigma \neq 0$:</u> Since charges of equal sign repel they tend to concentrate on the surface of the conductor. On the surface the charge density may vary and tends to increase near edges.

<u>$\mathbf{E} = 0$ inside the conductor:</u> Consider any point inside the conductor. Due to the fact

that $\rho = 0$ we have from Gauss' law that the r.h.s. of (2.17) equals zero for all surfaces surrounding this point no matter what shape they have or how small they are. Hence the electric field has to vanish at all points inside the conductor. A more physical argument goes as follows: Having $\mathbf{E} \neq 0$ inside the conductor would cause movement of charges in its interior. Hence it would not be possible to have electrostatics which nevertheless is observed in Nature.

E is perpendicular to the surface of the conductor: A non-vanishing component of the electric field on the surface would cause movement of charges on its surface until equilibrium is reached.

A conductor is an equipotential surface: Take two points **a** and **b** on the surface of the conductor and a path C connecting them on the surface. Then from (2.9) the potential difference is $\Phi(\mathbf{b}) - \Phi(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{r} = 0$, since the displacement is perpendicular to the electric field. Since the chosen points are arbitrary that proves that all surface points have the same potential.

Conductors in contact: When two conductors are connected by another conductor (a metal wire) they eventually acquire the same potential. This is achieved by positive (negattive) charges flowing from the conductor with the higher (lower) potential to that with lower (higher). The time it takes to achieve that, the new common potential and the amount of charge exchanged depends on the material, the shape of the conductors and what their relative position is.

Grounded conductor: The Earth is an equipotential surface and by convention has $\Phi = 0$. A conductor connected to Earth has also zero potential and it is called *grounded* (since the Earth is considered infinitely large).

Exercise: Consider two conducting spheres of radii R_i and charges Q_i, $i = 1, 2$. If we connect them by a long very thin wire show that they will acquire a common potential Φ' and charges Q_i' given by

$$Q_i' = \frac{R_i}{R_1 + R_2}(Q_1 + Q_2)\ , \quad i = 1, 2\ , \qquad \Phi' = \frac{Q_1 + Q_2}{R_1 + R_2}\ . \tag{2.40}$$

2.6 Electrostatic potential energy and pressure

We would like to find the energy associated with a static electric field. We start by writing down the analog of (2.36) for a continues distribution of charges, i.e.

$$\boxed{W = \frac{1}{2}\int_V dv\rho(\mathbf{r})\Phi(\mathbf{r})}\ . \tag{2.41}$$

To proceed we first trivially extend the integration over the entire space $\mathbb{R}^3$. Using (2.15) we may replace the density ρ by $-\frac{1}{4\pi}\nabla^2\Phi$. Then by partially integrating (note that since the integration was extended to all space the surface term vanishes as all fields vanish at spatial infinity) and using the definition (2.9) we express the energy solely in terms of the electric field as

$$\boxed{W = \frac{1}{8\pi}\int_{\mathbb{R}^3} dvE^2}\ . \tag{2.42}$$

Comment: One easily notices that whereas (2.42) is definitely positive its original version (2.36) can have either sign. The reason for this apparent paradox is that (2.36) does not actually take into account the amount of work necessary to make a point charge. In fact, the energy associated with the electric field of a point change is found using (2.42) to diverge (**exercise**). A point charge is already given and cannot be taken apart. Hence, it does not make sense to ask for the amount of work it took to make it. The reason for this discrepancy is that in (2.36) the potential at $\mathbf{r}_i$ as given by $\Phi(\mathbf{r}_i)$ is due to all charges except the one in that position whereas in (2.42) $\Phi(\mathbf{r})$ is the full potential. In a continuous distribution there is no real distinction since the amount of charge at a single point is vanishingly small. Hence, for electric fields due to discrete distributions of charges we better use (2.36) to evaluate the total potential energy.

We would like to compute the pressure exerted on the surface of a conductor with charge density on its surface σ. The force on a small area δS with charge density σ is due to the charge in the rest of the conductor. Let's denote that field by $\mathbf{E}_{\text{other}}$ which should have continuous components across the surface of the conductor. On the other hand the electric field due to the small area is given by (2.25). Hence, by superimposing the two contributions we have that, the field just above and just below the surface of the conductor is

$$\mathbf{E}_2 = \mathbf{E}_{\text{other}} + 2\pi\sigma\hat{\mathbf{n}} \,,$$
$$\mathbf{E}_1 = \mathbf{E}_{\text{other}} - 2\pi\sigma\hat{\mathbf{n}} \,. \tag{2.43}$$

Since the electric field inside the conductor is zero we have that $\mathbf{E}_1 = 0$, which implies that $\mathbf{E}_{\text{other}} = 2\pi\sigma\hat{\mathbf{n}}$. Since this small surface area carries charge $\delta q = \sigma\delta S$ the force on it is $\mathbf{E}_{\text{other}}\delta q = 2\pi\sigma^2\delta S$ and is directed upwards. Dividing by δS we find that the pressure is

$$\boxed{\mathbf{P} = 2\pi\sigma^2\hat{\mathbf{n}}} \,. \tag{2.44}$$

The total force is found by integrating over the surface, i.e.

$$\mathbf{F} = \int_S \mathbf{P} dS \,. \tag{2.45}$$

Finally, it is important to know the energy associated with an *external* charge distribution. In that case the factor $\frac{1}{2}$ in (2.41) is not necessary and we have that

$$\boxed{W_{\text{ext}} = \int_V dv \rho(\mathbf{r})\Phi(\mathbf{r})} \,. \tag{2.46}$$

2.7 Multipole expansion

In materials there are strong field fluctuations at the atomic scales of the order $10^{-9}m$. However, the observation points are typically at distances of about $10^{-3}m$. Mathematically this implies that in the expression for the potential (2.10) due to a charge distribution $\rho(\mathbf{r})$ we have that $r \gg r'$. Recall next the Taylor expansion of a function

$$f(\mathbf{r}) = f(\mathbf{0}) + x_i\partial_i f|_{\mathbf{r}=\mathbf{0}} + \frac{1}{2}x_i x_j \partial_i \partial j f|_{\mathbf{r}=\mathbf{0}} + \cdots \,, \tag{2.47}$$

where we recall Einstein's summation convention over repeated indices. Applying this for $f(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$ we obtain

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{1}{r^3}x'_i x_i + \frac{1}{2r^5}x'_i x'_j(3x_i x_j - r^2\delta_{ij}) + \cdots \,. \tag{2.48}$$

Then the potential (2.10) can be expanded as

$$\Phi(\mathbf{r}) = \frac{q}{r} + \frac{\mathbf{P}\cdot\mathbf{r}}{r^3} + \frac{1}{2r^5}Q_{ij}x_i x_j + \cdots \ , \tag{2.49}$$

where

$$q = \int dv \rho(\mathbf{r}') \ , \tag{2.50}$$

is the total charge or *electric monopole moment*. The vector

$$\boxed{\mathbf{P} = \int d^3\mathbf{r}'\rho(\mathbf{r}')\mathbf{r}'} \ , \tag{2.51}$$

is called the *electric dipole moment*.[1] Finally

$$\boxed{Q_{ij} = \int d^3\mathbf{r}'\rho(\mathbf{r}')(3x'_i x'_j - r'^2\delta_{ij})} \ , \tag{2.52}$$

is called the *electric quadrupole moment*. Note that Q_{ij} is symmetric and traceless.

A configuration of charges discrete or continuous is called a *dipole* is it net total charge vanishes. An elementary dipole is formed by two opposite charges q and $-q$ a distance d apart. Using (2.51) we find that the magnitude of the dipole moment is $P = qd$ and is directed towards the positive charge. Similarly, a configuration of charges is called a *quadrupole* if it has vanishing charge and dipole moments. Analogous names are reserved for higher moments.

Exercise: Consider four charges $+q$,$-q$, $+q$ and $-q$ placed in that order at the vertices of a square of edge a. Show that the dipole moment is zero and that the quadrupole moment's only non-vanishing component is $Q_{xy} = 3qa^2$.

Exercise: a) Show that the dipole moment is independent of the origin of the coordinate system it is computed in, provided that the monopole moment vanishes.
b) Similarly, show that the quadrupole moment is independent of the origin of the coordinate system it is computed in, provided that the monopole and dipole moments both vanish.
c) Argue that this is true for the mth moment as well provided that all the previous ones vanish.

[1] Not to be confused with the pressure that is denoted in (2.44) by the same symbol.

2.7.1 Field and energy of an electric dipole

To compute the electric field associated with an electric dipole we first isolate the corresponding potential

$$\Phi_{\text{dip}}(\mathbf{r}) = \frac{\mathbf{P}\cdot\mathbf{r}}{r^3} = -P_i\partial_i\frac{1}{r} \tag{2.53}$$

and then we use the definition (2.9). We have that

$$E_i = P_j\partial_i\partial_j\frac{1}{r} = -\frac{1}{r^3}\left(P_i - 3\frac{\mathbf{P}\cdot\mathbf{r}}{r^2}x_i\right) - \frac{4\pi}{3}P_i\delta^{(3)}(\mathbf{r})\ . \tag{2.54}$$

where we have used (E.20). Hence,

$$\mathbf{E} = \frac{3(\mathbf{P}\cdot\hat{\mathbf{r}})\hat{r} - \mathbf{P}}{r^3} - \frac{4\pi}{3}\mathbf{P}\delta^{(3)}(\mathbf{r})\ . \tag{2.55}$$

To compute the energy associated with charge-dipole and dipole-dipole interactions we first expand the potential as

$$\begin{aligned}\Phi(\mathbf{r}) &= \Phi(\mathbf{0}) + x_i\partial_i\Phi|_{\mathbf{r}=\mathbf{0}} + \frac{1}{2}x_ix_j\partial_i\partial j\Phi|_{\mathbf{r}=\mathbf{0}} + \ldots\ . \\ &= \Phi(\mathbf{0}) - x_iE_i(\mathbf{0}) - \frac{1}{2}x_ix_j\partial_iE_j|_{\mathbf{r}=\mathbf{0}} + \ldots\ .\end{aligned} \tag{2.56}$$

If the electric field is due to some external sources then $\partial_iE_i|_{\mathbf{r}=\mathbf{0}} = 0$. Then

$$\Phi(\mathbf{r}) = \Phi(\mathbf{0}) - x_iE_i(\mathbf{0}) - \frac{1}{6}(3x_ix_j - r^2\delta_{ij})\partial_iE_j|_{\mathbf{r}=\mathbf{0}} + \ldots\ . \tag{2.57}$$

Thus the energy associated with external sources (2.46) is

$$W_{\text{ext}} = q\Phi(\mathbf{0}) - \mathbf{P}\cdot\mathbf{E}|_{\mathbf{r}=\mathbf{0}} - \frac{1}{6}Q_{ij}\partial_iE_j|_{\mathbf{r}=\mathbf{0}} + \cdots\ , \tag{2.58}$$

where I used the definitions (2.51) and (2.52). Hence, a charge will interact with the potential, a dipole with the electric field, a quadrupole with the gradient of the electric field and so on.

Exercise: Show that the energy associated with the interaction of a charge q at $\mathbf{r_1}$ and a dipole $\mathbf{P}$ at $\mathbf{r_2}$ is

$$W_{q-P} = q\frac{\mathbf{P}\cdot(\mathbf{r_1} - \mathbf{r_2})}{|\mathbf{r_1} - \mathbf{r_2}|^3}\ . \tag{2.59}$$

Exercise: Show that the energy associated with the interaction of a dipole $\mathbf{P_1}$ at $\mathbf{r_1}$ with another dipole at $\mathbf{P_2}$ at $\mathbf{r_2}$ is

$$W_{P-P} = \frac{\mathbf{P_1}\cdot\mathbf{P_2} - 3(\mathbf{P_1}\cdot\hat{\mathbf{n}})(\mathbf{P_2}\cdot\hat{\mathbf{n}})}{|\mathbf{r_1}-\mathbf{r_2}|^3} + \frac{4\pi}{3}\mathbf{P_1}\cdot\mathbf{P_2}\delta^{(3)}(\mathbf{r_1}-\mathbf{r_2})\ , \qquad (2.60)$$

where $\hat{\mathbf{n}} = \frac{\mathbf{r_1}-\mathbf{r_2}}{|\mathbf{r_1}-\mathbf{r_2}|}$ is the unit vector from the dipole 2 to dipole 1. This energy can be positive or negative depending on the orientation of the dipoles. In many instances, in particular in quantum mechanical applications of the above expression, it is only the δ-function term that contributes.

3 Boundary value problems in electrostatics

Our aim is to solve Poisson equation (2.15) in various cases of interest where boundary effects play an important role. There are several methods to do that some of which we present below. Before we do that we comment on the uniqueness of the solution.

Uniqueness theorem 1: Consider a region in space with volume V free of charges and with a boundary surface S. Specifying the b.c. on S completely determines the potential Φ.

Proof: Assume two different solutions Φ_i, $i = 1,2$ satisfying the Laplace eq. and the same b.c. on the surface. Then their difference $\Phi = \Phi_1 - \Phi_2$ satisfies the Laplace eq. as well but with zero b.c. on the surfaces. It is a fact that all extrema of the Laplace eq. occur at the boundaries. Hence the maximum and minimum of Φ are both zero. Therefore, $\Phi = 0$ everywhere and $\Phi_1 = \Phi_2$, i.e. the solution is unique.

Uniqueness theorem 2: Consider a region in space with volume V having charges with density ρ inside, which also contains several conductors with surfaces S_a, $a = 1,2,\ldots$. There can also be an outer boundary surface S surrounding the volume which could be taken to infinity. The electric field is completely determined if the total charges in each conductor are specified.

Proof: Assume two different electric fields $\mathbf{E_1}$ and $\mathbf{E_2}$, both satisfying (2.14) for the same ρ and having the same total charges on each surface S_a (that includes the possible outer surface). Let's also denote the corresponding potentials by Φ_i, $i = 1,2$ and their values on the conducting surfaces $\Phi_{a,i}$. These values are constant since the boundaries are equipotential surfaces. We emphasize that the way charge is distributed could be different and is characterized by charge surface densities $\sigma_{a,i}$.

The difference $\mathbf{E} = \mathbf{E_1} - \mathbf{E_2}$ satisfies (2.14) with zero r.h.s and the potential is $\Phi = \Phi_1 - \Phi_2$. On the boundaries the surface charge densities are $\sigma_a = \sigma_{a,1} - \sigma_{a,2}$, but in each one of them the total charge is zero. Finally, the potential is $\Phi_a = \Phi_{a,1} - \Phi_{a,2}$. Consider first

$$\nabla \cdot (\Phi \mathbf{E}) = \Phi \underbrace{\nabla \cdot \mathbf{E}}_{0} + \mathbf{E} \cdot \underbrace{\nabla \Phi}_{-\mathbf{E}} = -E^2 \,. \tag{3.1}$$

Hence, integrating over the volume in question we obtain

$$-\int_V dv E^2 = \int_V dv \nabla \cdot (\Phi \mathbf{E}) = \sum_a \oint_{S_a} dS \hat{\mathbf{n}} \cdot (\Phi \mathbf{E}) = \sum_a \Phi_a \underbrace{\oint_{S_a} dS \hat{\mathbf{n}} \cdot \mathbf{E}}_{0} = 0 \tag{3.2}$$

where in the sums we wave included the outer boundary and where we have also pulled out of the integral Φ_a since on the S_a it is just a constant. The last zero comes from the fact that the total charge in each surface is zero. Hence $\mathbf{E} = 0$ everywhere and $\mathbf{E_1} = \mathbf{E_2}$, i.e. the solution is unique.

3.1 Method of images

Sometimes it is possible, due to the symmetry of a problem to use *image charges*. These should be situated outside the space in which we would like to solve the Poisson equation so that they do not alter the charge density. We stretch that it is possible to satisfy the b.c. without image charges. However, it is very convenient to use these fictitious charges solely as a computational tool.

3.1.1 Charge over a conducting grounded infinite plane

The canonical example is to consider a charge q at a distance d from a infinitely extended grounded conducting plane as in the Fig. 4. In order to satisfy the zero potential b.c. at $z = 0$ consider the mirror image charge $-q$.
Then the potential is given by

$$\Phi(\rho, z) = \frac{q}{\sqrt{\rho^2 + (z-d)^2}} - \frac{q}{\sqrt{\rho^2 + (z+d)^2}} \,, \qquad \rho, z \geqslant 0 \,. \tag{3.3}$$

Then using (2.9) and Appendix D we compute the components of the electric field

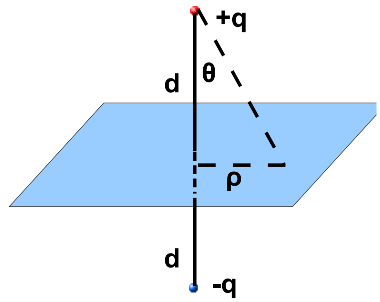

Figure 4: Charge q over an infinite conducting plane and its image.

(**exercise**)

$$E_\rho = q\frac{\rho}{(\rho^2+(z-d)^2)^{3/2}} - q\frac{\rho}{(\rho^2+(z+d)^2)^{3/2}} ,$$
$$E_z = q\frac{z-d}{(\rho^2+(z-d)^2)^{3/2}} - q\frac{z+d}{(\rho^2+(z+d)^2)^{3/2}} . \tag{3.4}$$

in cylindrical polar coordinates. Of course $E_\phi = 0$ indentically and that the tangential component of the electric field on the surface $z = 0$ vanishes, as it should be for a conductor, i.e. $E_\rho\big|_{z=0} = 0$.

The induced charge density on the surface is

$$\sigma(\rho) = \frac{1}{4\pi}E_z\big|_{z=0} = -\frac{q}{2\pi}\frac{d}{(\rho^2+d^2)^{3/2}} . \tag{3.5}$$

The total induced charge is (**exercise**)

$$Q_{\text{ind}} = \int_0^{2\pi} d\phi \int_0^\infty \rho d\rho \sigma(\rho) = \cdots = -q . \tag{3.6}$$

Hence it is opposite of the charge at $z = d$. This charge is supplied by the Earth to which the conductor is grounded so that it is always kept to zero potential.

By symmetry considerations, the force exerted by the plane on the charge has only z-component equal to

$$F_z = q\int_0^{2\pi} d\phi \int_0^\infty d\rho\rho\frac{\sigma(\rho)}{\rho^2+d^2}\cos\theta = -q^2d^2\int_0^\infty d\rho\frac{\rho}{(\rho^2+d^2)^3} = -\frac{q^2}{4d^2} . \tag{3.7}$$

This force is the same as if it is due to the image charge.

<u>**Exercise:**</u> Using (2.45) with charge density (3.5), compute the total force due exerted

on the conducting plane due to the charge pressure. You should find that it is finite and opposite to (3.7) even though the sheet is infinitely extended. This is in agreement with the action-reaction principle.

Remark: It is important to note that the energy of the electric field is not that of two opposite charges a distance $2d$ apart. The reason is that when we bring the charge q from infinity to the point $z = d$ it requires no extra effort to move its image. Hence, using (3.7) with d relaced by z and taking for convenience the straight path we compute that

$$W = -\int_{\infty}^{d} F_z dz = -\frac{q^2}{4d} , \qquad (3.8)$$

Exercise: Compute the potential of a charge q at a distance d over an infinite plane with the b.c. that the component normal to the plane of the electric field vanishes. This b.c. corresponds to keeping the surface free of charges. Compute also the potential on the surface as a function of the radial distance ρ (we refer to Fig. 4).

Exercise: A charge q is placed at a point as in the Fig. 5. The parts of the planes $x = 0$ and $y = 0$ marked with solid lines are grounded conducting planes extending indefinitely in the z direction.

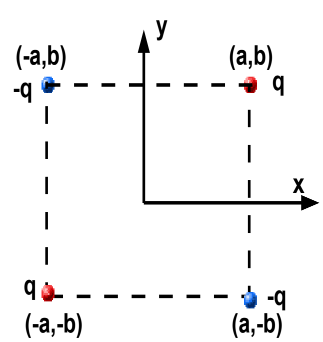

Figure 5: A charge q with coordinates $(x, y, z) = (a, b, 0)$ and its images.

Write down the b.c. specific to this problem and the solution for the potential. Compute the induced surface charge density on the conducting planes at $x = 0$ and at $y = 0$. Similarly, compute the line charge density on the line $x = y = 0$.

Exercise: If you are given one charge and a conducting plane how can you produce an electric field resembling that of a dipole? What extra do you need to produce an electric field resembling that of a quadrupole?

3.1.2 Charge over a conducting grounded sphere

As our next example we consider a grounded conducting sphere of radius R. We place a charge q at $\mathbf{r}_0 = (0, 0, d)$ with $d > R$ as in the Fig. 6.

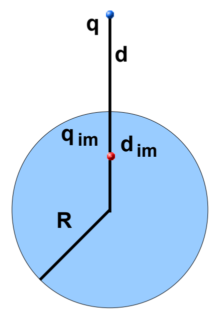

Figure 6: A charge q with $(x, y, z) = (0, 0, d)$ and its image q_{im} at $(0, 0, d_{\text{im}})$.

In this case it is not so obvious how to choose the image charge, in particular its magnitude and its position. What is certainly obvious is that it should be at a point in the line connecting the charge q and the center of the sphere. We write down an ansatz[2] of the form

$$\begin{aligned} \Phi(r, \theta) &= \frac{q}{|\mathbf{r} - \mathbf{r}_0|} + \frac{q_{\text{im}}}{|\mathbf{r} - \mathbf{r}_{\text{im},0}|} \\ &= \frac{q}{\sqrt{r^2 + d^2 - 2dr\cos\theta}} + \frac{q_{\text{im}}}{\sqrt{r^2 + d_{\text{im}}^2 - 2d_{\text{im}} r\cos\theta}} , \end{aligned} \tag{3.9}$$

where $\mathbf{r}_{\text{im},0} = (0, 0, d_{\text{im}})$ is the position of the image charge q_{im}, with $d > R$. Imposing that

$$\Phi(R, \theta) = 0 \quad \Longrightarrow \quad \frac{q}{\sqrt{R^2 + d^2 - 2dR\cos\theta}} + \frac{q_{\text{im}}}{\sqrt{R^2 + d_{\text{im}}^2 - 2d_{\text{im}} R\cos\theta}} = 0 , \forall\, \theta . \tag{3.10}$$

Obviously q_{im} has the opposite sign of q. Then we get the conditions

$$\frac{R^2 + d^2}{q^2} = \frac{R^2 + d_{\text{im}}^2}{q_{\text{im}}^2} , \qquad \frac{d}{q^2} = \frac{d_{\text{im}}}{q_{\text{im}}^2} . \tag{3.11}$$

In solving for d_{im} and q_{im} we should find that $d_{\text{im}} < R$ so that the charge density

[2] In physics and mathematics, an ansatz is an educated guess that is proved or confirmed later by its results.

outside the sphere is not altered. One finds that

$$q_{\rm im} = -\frac{R}{d} q\,, \qquad d_{\rm im} = \frac{R^2}{d}\,. \tag{3.12}$$

The charge on the surface is given by (**exercise**)

$$\sigma(\theta) = \frac{1}{4\pi} E_r\Big|_{r=R} = -q\frac{d^2-R^2}{4\pi R(d^2+R^2-2dR\cos\theta)^{3/2}}\,. \tag{3.13}$$

Exercise: Using (3.13) show that the force exerted by the sphere on the charge q is given by

$$\mathbf{F} = \frac{q q_{\rm im}}{(d-d_{\rm im})^2}\hat{z} = -\frac{q^2 R}{d^3}\left(1-\frac{R^2}{d^2}\right)\hat{z}\,. \tag{3.14}$$

Hence, similarly to the case of a charge above a conducting grounded plane in this case as well the force is as if the image is a real charge.

Exercise: Using (2.45) with the charge density (3.13) compute the total force due to charge pressure exerted on the conducting sphere. You should find that it is opposite to (3.14).

The problem we solved above helps to find the solutions to two seemingly more complicated variations of it:

Variation 1: Consider a conducting sphere of radius R with total charge Q which is kept isolated. As before we place a charge q at $\mathbf{r}_0 = (0,0,d)$ with $d > R$ as in the Fig. 6. The charge in the sphere is not distributed uniformly since it interacts with the charge q in such a way that the surface of the sphere is an equipotential surface. We may build the conditions of the problem in two steps. First we consider the case of the grounded sphere in which a total charge $-q\frac{R}{d}$ is induced on the surface of the sphere. Then we add extra charge equal to $Q + q\frac{R}{d}$ to the sphere which makes the total charge equal to Q which is precisely what is needed for our problem. Since the theory is linear and the superposition principle holds the potential is

$$\Phi(r,\theta) = \frac{q}{\sqrt{r^2+d^2-2dr\cos\theta}} - \frac{qR/d}{\sqrt{r^2+R^4/d^2-2R^2r/d\cos\theta}} + \frac{Q+qR/d}{r}\,. \tag{3.15}$$

Variation 2: Consider the same sphere and charge q at $\mathbf{r}_0 = (0,0,d)$ as before but now we keep its surface at constant potential V. Clearly the solution is

$$\Phi(r,\theta) = \frac{q}{\sqrt{r^2+d^2-2dr\cos\theta}} - \frac{qR/d}{\sqrt{r^2+R^4/d^2-2R^2r/d\cos\theta}} + \frac{VR}{r} . \qquad (3.16)$$

3.2 Method of separation of variables

This method is applicable when the potentials or the charge densities are specified on certain boundaries and we would like to determine the potential in the interior. An important ingredient is the choice of the appropriate coordinate system in which we will express the Laplace operator and b.c. This is based on the symmetries of the problem, i.e. geometry and b.c.

3.2.1 Cartesian coordinates

The method is better demonstrated by considering the two-dimensional Laplace eq.

$$(\partial_x^2 + \partial_y^2)\Phi = 0 , \qquad (3.17)$$

inside the rectangle of Fig. 7 with the b.c.

$$\Phi(0,y) = \Phi(a,y) = \Phi(x,b) = 0 , \qquad \Phi(x,0) = V(x) , \qquad (3.18)$$

where $V(x)$ is a given function.

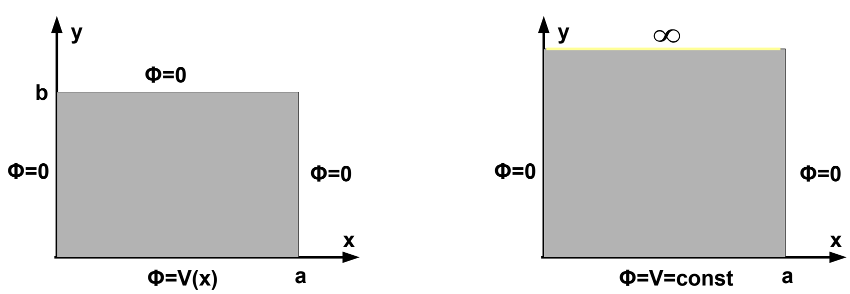

Figure 7: a) Rectangle with indicated b.c. b) Upper side is taken to infinity.

The method assumes that we may write the solution as $\Phi(x,y) = X(x)Y(y)$. Substituting into (3.17) we obtain

$$\frac{X''(x)}{X(x)} + \frac{Y''(x)}{Y(x)} = 0 . \qquad (3.19)$$

Since the two terms depend on different variables, the only way to satisfy this is to set both terms to opposite constants which are generally known as *separation of variables constants*. Let

$$\frac{X''(x)}{X(x)} = -k^2 \, , \qquad \frac{Y''(y)}{Y(y)} = k^2 \, , \tag{3.20}$$

where k is a real constant. The necessity for a negative sign in the first equation will become apparent shortly. In general, such and similar choices are made so that the b.c. can be satisfied. Returning to our case we find the solutions

$$X(x) = A \sin kx + B \cos kx \, , \qquad Y(y) = Ce^{ky} + De^{-ky} \, . \tag{3.21}$$

Imposing the b.c. at $x = 0$ we obtain $B = 0$. Similarly for $x = a$ we obtain $k = \frac{n\pi}{a}$, where n could be any integer. However, for $n = 0$ the solution becomes identically zero so that this choice is excluded. Moreover, the solutions with n and $-n$ are not linearly independent so that we may restrict to $n = 1, 2, \ldots$. Imposing the b.c. at $y = b$ we have that $Ce^{kb} + De^{-kb} = 0$. Hence the solution is of the form

$$\Phi_n(x, y) = A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a}(b - y) \, , \qquad n = 1, 2, \ldots \, . \tag{3.22}$$

The most general solution follows by superimposing the above solutions as

$$\Phi(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a}(b - y) \, , \tag{3.23}$$

where the A_n's are coefficients to be determined shortly. Imposing next the b.c. at $y = 0$ we have that

$$V(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} b \, . \tag{3.24}$$

Hence, using the orthogonality relation

$$\int_0^a dx \sin \frac{m\pi}{a} x \sin \frac{n\pi}{a} x = \frac{a}{2} \delta_{mn} \, . \tag{3.25}$$

we compute the coefficients

$$A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a dx V(x) \sin \frac{n\pi}{a} x \, . \tag{3.26}$$

Open rectangle: A particularly interesting case is when the rectangle is open on one side, say the upper one as depicted in Fig. 7. It is not necessary to repeat the computation. We just take the limit $b \to \infty$ in the above formulae. The result is (**exercise**)

$$\Phi(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x e^{-\frac{n\pi}{a}y} , \qquad B_n = \frac{2}{a}\int_0^a dx V(x) \sin\frac{n\pi}{a}x . \tag{3.27}$$

Consider next the important case in which $V(x) = V =$ const.. Then we compute that $B_n = \frac{4V}{n\pi}$ if n is odd and $B_n = 0$ if n is even. Hence

$$\Phi(x,y) = \frac{4V}{\pi}\sum_{n=1,3}^{\infty}\frac{1}{n}e^{-n\pi y/a}\sin\frac{n\pi}{a}x = \frac{2V}{\pi}\tan^{-1}\left(\frac{\sin\frac{\pi x}{a}}{\sinh\frac{\pi y}{a}}\right) . \tag{3.28}$$

With this expression is it very easy to show that the Laplace equation (3.17) as well as the b.c. (3.18) are indeed satisfied (**Exercise**). The infinite sum is computed by employing the result of the following exercise.

<u>Exercise:</u> Show that the infinite sum

$$\Sigma(x_1,x_2) = \sum_{n=1,3}^{\infty}\frac{e^{-nx_1}}{n}\sin nx_2 = \frac{1}{2}\tan^{-1}\left(\frac{\sin x_2}{\sinh x_1}\right) , \qquad x_1, x_2 > 0 . \tag{3.29}$$

Hint: First compute $\partial_{x_1}\Sigma$ which is nothing but an infinite geometric series and show that

$$\partial_{x_1}\Sigma = -\frac{1}{2}\frac{\sin x_2 \cosh x_1}{\sin^2 x_2 + \sinh^2 x_1} . \tag{3.30}$$

Integrating over x_1 and choosing appropriately the integration constant you should find (3.29).

<u>Exercise:</u> Show that the charge densities due to the potential in (3.28) at the surfaces $x = 0$ and $x = a$ are both given by

$$\sigma(y) = -\frac{V}{2\pi a}\frac{1}{\sinh\frac{\pi y}{a}} \tag{3.31}$$

and that at $y = 0$ by

$$\sigma(x) = \frac{V}{2\pi a}\frac{1}{\sin\frac{\pi x}{a}} . \tag{3.32}$$

In addition, verify that $E_x(x,0) = E_y(0,y) = E_y(a,y) = 0$.

3.2.2 Spherical coordinates

We will study the Laplace equation $\nabla^2\Phi = 0$ with the Laplace operator given in (D.7). In the separation of variables method we assume a dependence of the form

$$\Phi(r,\theta,\phi) = R(r)\Theta(\theta)\mathit{\Phi}(\phi) \ . \tag{3.33}$$

Upon substitution we find

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\mathit{\Phi}\sin^2\theta}\frac{d^2\mathit{\Phi}}{d\phi^2} = 0 \ . \tag{3.34}$$

The first term depends only on r and the second only on θ, but the third seems to depend on both θ and ϕ. However, if we assume that

$$\frac{d^2\mathit{\Phi}}{d\phi^2} + m^2\mathit{\Phi} = 0 \qquad \Longrightarrow \qquad \mathit{\Phi} \sim e^{im\phi} \ , \tag{3.35}$$

where m is a constant, there is no such dependence and in (3.34) the third term depends only on θ. Demanding periodicity in the azimuthal angle ϕ, i.e. $\mathit{\Phi}(\phi + 2\pi) = \mathit{\Phi}(\phi)$, gives the restriction $m \in \mathbb{Z}$. Naming for later convenience the separation of variables constant as $\ell(\ell+1)$ we obtain two equations, the so called radial equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R \tag{3.36}$$

and the angular equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0 \ . \tag{3.37}$$

The solution to the radial equation is easy to obtain. One finds (**Exercise**)

$$R(r) = Ar^\ell + \frac{B}{r^{\ell+1}} \ , \tag{3.38}$$

The appearance of these simple exponents explains the convenient choice of parametrization for the separation of variables constant made above.

In the following we will make the choice $m = 0$ which means we will study solutions with azimuthal symmetry. Then after changing the independent variable as $x = \cos\theta$, (3.37) becomes the so called *Legendre* differential equation. Its properties and solutions are summarized in Appendix F. What is more important is that for $\ell = 0, 1, 2, \ldots$ it admits polynomial solutions of order ℓ, called *Legendre Polynomials* $P_\ell(x)$. These form a complete set and obey the orthogonality relation

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'} \, . \tag{3.39}$$

Being a second order differential equation it also admits, for a given ℓ, a second independent non-polynomial solution labelled by $Q_\ell(x)$. However, these are singular at $x \pm 1$, i.e. at the poles of the sphere at $\theta = 0, \pi$, and, though useful in many applications, will not be considered here.

Putting everything together the most general, everywhere non-singular, with azimuthal symmetry solution of the Laplace equation in spherical coordinates is given by

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \, . \tag{3.40}$$

The constants A_ℓ and B_ℓ will be determined by the b.c. In particular, if $r = 0$ is part of the space then demanding finiteness of the solution requires that $B_\ell = 0$. Similarly, if the space extends to infinity and the elecric field vanishes there, then $A_\ell = 0$ except possibly for $\ell = 0$ since then the corresponding term is just a constant.

Example: Consider two concentric spheres with radii a and b with $b > a$. We would like to determine the potential for $a \leqslant r \leqslant b$ with b.c.

$$\Phi(a,\theta) = V_a(\theta) \, , \qquad \Phi(b,\theta) = V_b(\theta) \, , \tag{3.41}$$

for two given functions $V_a(\theta)$ and $V_b(\theta)$. From these, using (3.40) and (3.39) we find the algebraic system

$$\begin{aligned} r = a : \qquad & a^\ell A_\ell + \frac{B_\ell}{a^{\ell+1}} = (\ell + 1/2) \int_{-1}^{1} dx V_a(x) P_l(x) \equiv J_{a,\ell} \, , \\ r = b : \qquad & b^\ell A_\ell + \frac{B_\ell}{b^{\ell+1}} = (\ell + 1/2) \int_{-1}^{l} dx V_b(x) P_l(x) \equiv J_{b,\ell} \, . \end{aligned} \tag{3.42}$$

The solution to this is

$$A_\ell = \frac{a^{l+1} J_{a,\ell} - b^{l+1} J_{b,\ell}}{a^{2\ell+1} - b^{2\ell+1}} \, , \qquad B_\ell = \frac{a^\ell J_{b,\ell} - b^\ell J_{a,\ell}}{a^{2\ell+1} - b^{2\ell+1}} (ab)^{\ell+1} \, . \tag{3.43}$$

Hence, it only remains in specific applications to compute $J_{a,\ell}$ and $J_{b,\ell}$.

As limiting cases we have

$$\begin{aligned} b\to\infty: &\qquad \Phi(r,\theta)=\sum_{\ell=0}^{\infty} J_{a,\ell}\left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)\ , \qquad a\leqslant r<\infty\ ,\\ a\to 0: &\qquad \Phi(r,\theta)=\sum_{l=0}^{\infty} J_{b,\ell}\left(\frac{r}{b}\right)^{\ell} P_\ell(\cos\theta)\ , \qquad 0\leqslant r\leqslant b\ , \end{aligned} \tag{3.44}$$

corresponding to the field outside and inside a sphere.

Example: We will prove that the potential due to a point charge q located at the point $\mathbf{r}_0=(0,0,1)$ can be written as

$$\Phi(\mathbf{r})=q\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\theta)\ , \tag{3.45}$$

where $r_<$ ($r_>$) is the smaller (larger) between r and 1. In general we know that

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\frac{1}{\sqrt{r^2+1-2r\cos\theta}}=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{3.46}$$

and our task is to compute the coefficients A_ℓ and B_ℓ. Focusing on the z-axis with $\theta=0$ and using that $P_\ell(1)=1$ we find that

$$\frac{1}{|r-1|}=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)\ . \tag{3.47}$$

Taylor expanding the l.h.s.

$$\begin{aligned} r<1: &\quad \frac{1}{|r-1|}=\sum_{\ell=0}^{\infty} r^\ell \quad\Longrightarrow\quad A_\ell=1,\quad B_\ell=0\ ,\\ r>1: &\quad \frac{1}{|r-1|}=\sum_{\ell=0}^{\infty} r^{-\ell-1} \quad\Longrightarrow\quad A_\ell=0\ ,\quad B_\ell=1\ . \end{aligned} \tag{3.48}$$

Rearranging we find the given result. Note that for $t=r<1$ (3.45) reduces to the generating function for Legendre polynomials (F.11).

Exercise: Consider a uniform electric field $\mathbf{E_0}=E_0\hat{z}$ that fills up the entire space. Then place at the origin a conducting sphere of radius R.

a) Prove that then the potential is given by

$$\Phi(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta \,, \qquad r \geqslant R \tag{3.49}$$

and zero for $r \leqslant R$.

b) Show that outside the sphere this is the superposition of the potential due to the original electric field and that of a dipole with moment $\mathbf{P} = R^3 E_0 \hat{z}$.

c) Compute the induced surface charge and show that it produces an electric field that precisely cancels $\mathbf{E_0}$ inside the sphere.

4 Magnetostatics

We would like to study the fields of moving charges. A moving charge produces an electric field which is not given by Coulomb's expression (2.4). The latter is strictly valid for static charges and approximately valid for charges moving with velocities much smaller than the speed of light. In addition, even such charges should not accelerate much for Coulomb's law to be approximately valid. The exact expression is more involved and will be derived later.

Our focus in this chapter is to explore the fact that moving point charges or currents (composed by practically infinitely many point charges infinitesimally close) produce another kind of field called *magnetic*. As in electrostatics we will start with some experimentally observed law analogous to Coulomb law and work our way towards more general definitions and laws. In the presence of moving charges and time dependent currents both electric and magnetic fields can be present. Likely it was observed that wires carrying steady currents, that is time independent currents, exert forces on nearby charges provided that these are moving. On a single charge this is the *Lorentz force* and is given by

$$\mathbf{F} = \frac{q}{c}\mathbf{v} \times \mathbf{B} . \tag{4.1}$$

The time independent vector $\mathbf{B}$ depends on the configuration of the wires, i.e their shape or equivalently the mathematical curves or surfaces they form, and is proportional to the currents flowing in them. Note also that the magnetic force does not produce any work since $\mathbf{v} \cdot \mathbf{F} = 0$.

Finally, a few comments on the nature of the charge carriers are in order. This depends on the material. In metals, the charge carriers are the so called free electrons, i.e. electrons not bounded to a particular atom, but instead are able to move freely within the crystal structure of the metal. In electrolytes, such as salt water, the charge carriers are ions, atoms or molecules that have gained or lost electrons so they are electrically charged. In semiconductors (the material used to make electronic components, i.e., like transistors, integrated circuits etc), the charge carriers are electrons, and "holes" which are "travelling vacancies" of valence electrons which are bound to individual atoms, as opposed to conduction electrons. The holes effectively act as mobile positive charges.

The standard convention in the literature is to take as the direction of the current that of positively charged carriers. Hence, in metals the current convention is opposite to that of the actual motion of the charge carriers which are electrons.

Exercise: Consider a particle with charge q moving under the influence of the constant magnetic field $\mathbf{B} = B\hat{z}$. Show that its equations of motion reduce to

$$\dot{v}_x = -\omega v_y \,, \qquad \dot{v}_y = \omega v_x \,, \qquad \dot{v}_z = 0 \,, \qquad \omega \equiv \frac{qB}{mc} \tag{4.2}$$

and deduce that its motion is on a helix.

4.1 Biot–Savart law

It was found after much experimental and theoretical work that a wire carrying steady current I, defined as the charge passing through a point on the wire per unit time, produces a magnetic field given by

$$\mathbf{B} = \frac{I}{c}\int_C d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \,, \tag{4.3}$$

as in the Fig. 8.

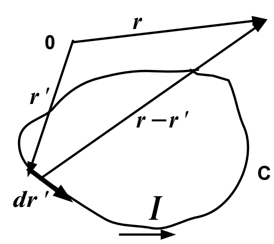

Figure 8: Contribution to the magnetic field of a wire loop carrying steady current I.

This is a particularly useful expression for the cases of electric currents through infinitesimally thin wires. However, in many application we have currents confined on surfaces or currents extended in all three dimensions.

In order to write down the magnetic field for a distribution of moving charges we introduce the concept of a volume density current. We will do that in the most general context which allows for time dependence. Consider a region in space with charge density $\rho(\mathbf{r}, t)$. The rate of changing of the total charge enclosed in this volume V is

$$-\frac{dq}{dt} = -\int_V dv \frac{\partial \rho}{\partial t} \,. \tag{4.4}$$

From charge conservation, this should equal the charge leaving from the surface S bounding this volume (assuming that (4.4) is negative). By inspecting the elementary volume element in Fig. 9, the net charge flowing out of it per unit time is $\rho dS\hat{\mathbf{n}} \cdot \mathbf{v}$.

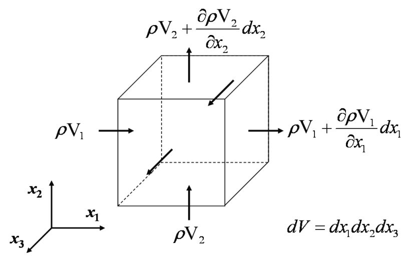

Figure 9: Charge flowing in and out of an infinitesimal cube.

Then, the total charge leaving the volume V is given by

$$\oint_S dS\rho\hat{\mathbf{n}} \cdot \mathbf{v} = \oint_V dv \nabla \cdot \mathbf{J} \,, \tag{4.5}$$

from the divergence theorem, where

$$\mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t) \,, \tag{4.6}$$

is called the *current density*. It measures the amount of charge crossing an area per unit time. Equating the two expressions and recalling that the volume V is arbitrary and in particular that it can be made arbitrarily small, we arrive at the *continuity equation*

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0} \,. \tag{4.7}$$

The *steady state* of a distribution of charges and currents is defined as

$$\frac{\partial \rho}{\partial t} = 0 \quad \Longrightarrow \quad \nabla \cdot \mathbf{J} = 0 \,. \tag{4.8}$$

Hence, we see that we can have time independent current densities which defines *magnetostatics*.

4.2 Equations for magnetostatics

An obvious generalization of Biot–Savart's law (4.3) when we have volume distributions of currents is

$$\mathbf{B} = \frac{1}{c}\int d^3\mathbf{r}'\, \mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\ . \tag{4.9}$$

This is analogous to (2.6) for electrostatics and as in that case we would like to discover the differential equations obeyed by $\mathbf{B}$. As in the electrostatics case we will examine, due to the Helmholtz theorem, the divergence and the curl of $\mathbf{B}$. We have that

$$B_i = \frac{1}{c}\epsilon_{ijk}\int_V d^3\mathbf{r}' J_j(\mathbf{r}')\ \underbrace{\frac{x_k - x'_k}{|\mathbf{r}-\mathbf{r}'|^3}}_{-\partial_k(1/|\mathbf{r}-\mathbf{r}'|)}\ , \tag{4.10}$$

which implies that

$$\boxed{\mathbf{B} = \nabla\times\mathbf{A}}\ , \tag{4.11}$$

where

$$\mathbf{A} = \frac{1}{c}\int d^3\mathbf{r}'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{4.12}$$

is called the *vector potential*. Hence, immediately we arrive at

$$\boxed{\nabla\cdot\mathbf{B} = 0}\ . \tag{4.13}$$

In addition from (B.12)

$$\nabla\times\mathbf{B} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}\ . \tag{4.14}$$

However, from (4.12) we have that

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{1}{c}\int d^3\mathbf{r}' J_i(\mathbf{r}')\partial_i\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\frac{1}{c}\int d^3\mathbf{r}'\ J_i(\mathbf{r}')\partial'_i\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\\ &= \frac{1}{c}\int d^3\mathbf{r}\ \partial'_i J_i(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 0\ ,\end{aligned} \tag{4.15}$$

where we've performed a partial integration and have disregarded a boundary term assuming that the currents drop off sufficiently fast at spatial infinity. The last line follows from the fact that $\partial_i J_i(\mathbf{r}) = 0$ since we are dealling with magnetostatics. The last term is

$$\nabla^2 \mathbf{A} = \frac{1}{c}\int d^3\mathbf{r}'\, \mathbf{J}(\mathbf{r}')\, \underbrace{\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{-4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}')} \ . \tag{4.16}$$

Hence

$$\boxed{\nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J}} \tag{4.17}$$

and

$$\boxed{\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}} \ . \tag{4.18}$$

Equations (4.13) and (4.18) are the basic equations of magnetostatics and are analogous to (2.12) and (2.14) for electrostatics. The vanishing of the r.h.s. of (4.13) implies that the are no free magnetic charges. Hence, the flux of any magnetic field over a closed surface vanishes, i.e.

$$\boxed{\oint_S \mathbf{B}\cdot\hat{\mathbf{n}} dS = 0} \ , \tag{4.19}$$

By using Stokes' theorem in (4.18) we easily find that

$$\boxed{\int_S (\nabla\times\mathbf{B})\cdot\hat{\mathbf{n}} dS = \oint_C \mathbf{B}\cdot d\mathbf{r} = \frac{4\pi I}{c}} \ , \tag{4.20}$$

where the total current through the surface is

$$I = \int_S \mathbf{J}\cdot\hat{\mathbf{n}} dS \ . \tag{4.21}$$

That means that the circulation of a magnetic field over a closed curve C is proportional to the total current I passing through the surface. This is the *Ampere's law*.

As in the case of the Gauss's law for electrostatics, Ampere's law can be used to find the magnetic field in cases of high symmetry.

It is amusing to note the formal correspondence between fields and symbols

$$\boxed{\mathbf{E} \leftrightarrow \mathbf{B}\ , \qquad \cdot \leftrightarrow \times\ , \qquad \rho \leftrightarrow \frac{\mathbf{J}}{c}\ , \qquad \Phi \leftrightarrow \mathbf{A}} \ . \tag{4.22}$$

Exercise: Show that the magnetic field of a long straight wire carrying a steady current I is

$$\mathbf{B} = \frac{2I}{c\rho}\hat{\phi}\ , \tag{4.23}$$

in polar coordinates as in the Fig. 10.

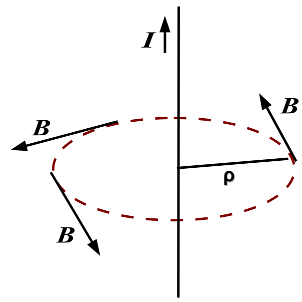

Figure 10: The magnetic field of a very long wire carrying steady current I.

There are, at least, two methods to compute this. One is by using Biot–Savart law and another by using Ampere's law and symmetry arguments.

Evidently, magnetic forces are exerted between wires carrying currents because these are made out of moving charges.

Exercise: Let two very long parallel wires carrying steady currents I_1 and I_2 at a distance ρ apart. The force per unit length exerted by wire 1 to wire 2 is given by

$$\mathbf{F}_{12} = -\frac{2I_1 I_2}{c\rho}\hat{\rho} \,. \tag{4.24}$$

Therefore, two wires carrying current in the same direction attract each other, and they repel if the currents are opposite in direction.

Exercise: Show that the magnetic field of a straight long wire (4.23) satisfies (4.13) and (4.18) with $\mathbf{J} = I\delta^{(2)}(\mathbf{r})$.

Exercise: An infinite current sheet consists of an infinite number of parallel wires all carrying the same current I. Assume that there are N wires per unit length. Find the magnetic field as in Fig. 11.

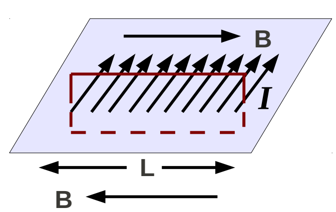

Figure 11: The magnetic field of an infinite sheet carrying surface current.

Using Ampere's law for the rectangle indicated in the figure and symmetry arguments show that the magnitude of the magnetic field is given by

$$B = \frac{2\pi I}{c} N , \tag{4.25}$$

it is parallel to the infinite sheet and perpendicular to the current flowing in the wires. Note that its direction has a discontinuity as we cross the sheet. Arrive at the same result by using the superposition of the magnetic field of the infinite number of wires by using (4.23).

Exercise: Consider a circular loop of radius R carrying steady current I. Show that the magnetic field in the symmetry axis at a distance z from the center of the loop is given by

$$\mathbf{B} = \frac{2\pi}{c} \frac{R^2 I}{(R^2 + z^2)^{3/2}} \hat{z} . \tag{4.26}$$

Remarks:

• One might be tempted to use (4.9) in order to find the magnetic field of a single point charge. Evidently this cannot be possible since a point charge cannot produce a steady current. Nevertheless, let's proceed and take, using (4.6) for the current charge density due to a charge at $\mathbf{r} = \mathbf{0}$ that $\mathbf{J} = q\mathbf{v}\delta^{(3)}(\mathbf{r})$. Then one finds

$$\mathbf{B} = \frac{q}{c} \frac{\mathbf{v} \times \mathbf{r}}{r^3} . \tag{4.27}$$

This formula is in general wrong. It is valid only if the so called *retardation effects* we will examine later can be neglected. An essential condition for that is that the velocities and the acceleration of charges are small.

Exercise: Show that (4.27) satisfies (4.13) but not (4.18).

• The choice of the vector potential is by no means unique. Consider a transformation replacing a given $\mathbf{A}$ by

$$\mathbf{A} \to \mathbf{A} + \nabla\Psi , \tag{4.28}$$

where Ψ is an arbitrary function. Then clearly, we see from (4.11) that the magnetic field is left invariant, i.e. does not change. The above transformation is a particular type of what is known as *gauge transformation*, a concept of immense importance in Theoretical and Mathematical Physics. This freedom in choosing the vector potential $\mathbf{A}$ allows one to choose one expression involving $\mathbf{A}$ (and possibly derivatives on it) arbitrarily. This is called a *gauge choice* or a *gauge fixing*. In practice we choose the most convenient form to perform our computations. In our case a convenient choice is $\nabla \cdot \mathbf{A} = 0$ which is called the *Coulmb gauge*. This is obeyed by the vector potential for magnetic currents in free space, i.e. with no boundaries as it was shown in (4.15).

Note that this is not exactly analogous to the situation with the potential Φ in electrostatics. In that case the function that we could shift with had to obey the Laplace equation (see the discussion below (2.15)) whereas in here it is arbitrary. What is analogous with the electrostatic case is the fact that in here as well in order to stay within the Coulomb gauge, a further gauge variation by a function Ψ in (4.28) has to satisfy the Laplace eq.

• In a region of space where there are charge $\rho(\mathbf{r})$ and current $\mathbf{J}(\mathbf{r})$ densities the total force they experience due to external electric $\mathbf{E}(\mathbf{r})$ and magnetic $\mathbf{B}(\mathbf{r})$ fields is

$$\mathbf{F} = \int_V d^3\mathbf{r} \left(\rho(\mathbf{r})\mathbf{E}(\mathbf{r}) + \frac{1}{c}\mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \right) . \tag{4.29}$$

Similarly the total torque is

$$\mathbf{N} = \int_V d^3\mathbf{r}\, \mathbf{r} \times \left(\rho(\mathbf{r})\mathbf{E}(\mathbf{r}) + \frac{1}{c}\mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \right) . \tag{4.30}$$

4.2.1 Magnetic field components across a surface carrying current

To find the appropriate b.c. for the magnetic field on a surface carrying currents we perform manipulations similar to those we have presented for the electric field. We find that these lead to

$$\boxed{(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} = 0} . \tag{4.31}$$

Hence, the component of the magnetic field normal to the surface is continuous along the surface. Similarly, for the tangential component

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) = \frac{4\pi}{c}\mathbf{K}} , \tag{4.32}$$

where $\mathbf{K}$ is the surface density current. A discontinuity of the tangential component of the magnetic field when crossing a surface, necessarily induces a surface density current on it.

These b.c. are illustrated in Fig. 12.

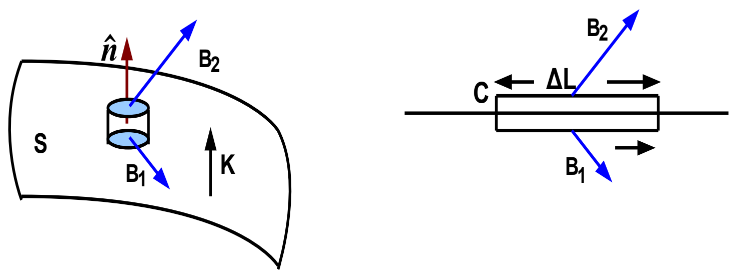

Figure 12: a) The continuity of the normal component of the magnetic field across a surface. b) The discontinuity of the tangential component of the magnetic field across a surface with density current **K**. The normal $\hat{\mathbf{n}}$ points from side 1 to 2.

Note that the vector potential **A** across a surface with a current density is continuous, since a discontinuity would give rise to a magnetic field with a delta-function profile (see Appendix E).

4.3 Multipole expansion

Consider a distribution of currents localized in space. Using (2.48) we may derive the following expansion of the vector potential (4.12) for $r \gg r'$

$$A_i(\mathbf{r}) = \frac{1}{cr}\int d^3\mathbf{r}' J_i(\mathbf{r}') + \frac{1}{cr^3}\mathbf{r}\cdot\int d^3\mathbf{r}'\mathbf{r}' J_i(\mathbf{r}') + \cdots \,. \tag{4.33}$$

In order to manipulate further these two terms note first the identity

$$\int d^3\mathbf{r}(f\mathbf{J}\cdot\nabla g + g\mathbf{J}\cdot\nabla f) = 0\,, \tag{4.34}$$

where f and g are any two functions that fall off sufficiently fast away from the current sources. This identity can be proven by integration by parts and use of $\nabla\cdot\mathbf{J} = 0$.

Choosing in (4.34) $f = 1$ and $g = x_i$ we arrive at

$$\int d^3\mathbf{r}' J_i(\mathbf{r}') = 0\,, \tag{4.35}$$

which shows that the first term in the expansion (4.33) that would have corresponded to a magnetic charge is absent. Choosing next in (4.34) $f = x_i$ and $g = x_j$ we arrive at

$$\int d^3\mathbf{r}'(x_i' J_j + x_j' J_i) = 0\,. \tag{4.36}$$

Using this

$$x_j \int x'_j J_i = \frac{1}{2} x_j \int (x'_j J_i - x'_i J_j) = -\frac{1}{2} x_j \epsilon_{ijk} \int (\mathbf{r}' \times \mathbf{J})_k = -\frac{1}{2} (\mathbf{r} \times \int \mathbf{r}' \times \mathbf{J})_i \,, \qquad (4.37)$$

where in the manipulations for convenience we dropped the volume element $d^3\mathbf{r}'$ and also note that the current components depend on $\mathbf{r}'$. Putting everything together

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m} \times \mathbf{r}}{r^3} + \cdots \,, \qquad (4.38)$$

where

$$\boxed{\mathbf{m} = \frac{1}{2c} \int d^3\mathbf{r}\, \mathbf{r} \times \mathbf{J}(\mathbf{r})} \,, \qquad (4.39)$$

is called the *magnetic moment*. The density of magnetic moment

$$\mathbf{M}(\mathbf{r}) = \frac{1}{2c} \mathbf{r} \times \mathbf{J}(\mathbf{r}) \,. \qquad (4.40)$$

is called *magnetization*.

Exercise: Consider a closed wire carrying current I. Assume also that it is confined on the $x-y$ plane and that it encloses a total area S. Show the the magnetic moment is

$$\mathbf{m} = \frac{IS}{c} \hat{z} \,. \qquad (4.41)$$

Exercise: Show that the magnetic field of a magnetic dipole is

$$\boxed{\mathbf{B} = \frac{3(\mathbf{m} \cdot \hat{r})\hat{\mathbf{r}} - \mathbf{m}}{r^3} + \frac{8\pi}{3} \mathbf{m} \delta^{(3)}(\mathbf{r})} \,. \qquad (4.42)$$

The manipulations are analogous to those that led for an electric dipole field to (2.55).

4.3.1 Force, torque and energy of magnetic moments in external fields

Consider a current distribution $\mathbf{J}(\mathbf{r})$ localized in the region in space around $\mathbf{r} = 0$ and apply to it an external magnetic field $\mathbf{B}(\mathbf{r})$, i.e. one due to other sources. We would like to obtain expressions for the force and torque that it experiences. Assuming that $\mathbf{B}(\mathbf{r})$ is well behaved and slow varying near $\mathbf{r} = 0$ we have the expansion

$$B(\mathbf{r}) = B_i(\mathbf{0}) + (\mathbf{r} \cdot \nabla B_i(\mathbf{0})\big|_{\mathbf{r}=0} + \cdots \,. \qquad (4.43)$$

Exercise: Using (4.29) and (4.30) and the above expansion show that the force and torque experienced by the distribution are given by

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})\big|_{\mathbf{r}=0} + \cdots = (\mathbf{m}\cdot\nabla)\mathbf{B}(\mathbf{0}) + \cdots \tag{4.44}$$

and

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}(\mathbf{0}) + \ldots . \tag{4.45}$$

The above expression for the force defines the interaction energy between a magnetic dipole and a magnetic field. This is extracted by writing $\mathbf{F} = -\nabla W_{\rm dip}$, where

$$W_{\rm dip} = -\mathbf{m}\cdot\mathbf{B} . \tag{4.46}$$

This is a very important expression and has applications in atomic Physics.

Exercise: Show that the energy associated with the interaction of a magnetic dipole $\mathbf{m_1}$ at $\mathbf{r_1}$ with another magnetic dipole at $\mathbf{m_2}$ at $\mathbf{r_2}$ is

$$W_{\rm dip-dip} = \frac{\mathbf{m_1}\cdot\mathbf{m_2} - 3(\mathbf{m_1}\cdot\hat{\mathbf{n}})(\mathbf{m_2}\cdot\hat{\mathbf{n}})}{|\mathbf{r_1}-\mathbf{r_2}|^3} - \frac{8\pi}{3}\mathbf{m_1}\cdot\mathbf{m_2}\delta^{(3)}(\mathbf{r_1}-\mathbf{r_2}) , \tag{4.47}$$

where $\hat{\mathbf{n}} = \frac{\mathbf{r_1}-\mathbf{r_2}}{|\mathbf{r_1}-\mathbf{r_2}|}$ is the unit vector from the dipole 2 to dipole 1. This energy can be positive or negative depending on the orientation of the dipoles.

5 Time varying fields and Maxwell's equations

We start now to consider the electric and magnetic fields produced by time dependent sources, i.e. charges and/or currents. Let a surface S with boundary C moving with constant velocity $\mathbf{v}$ in a region of space with time dependent, in general, electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

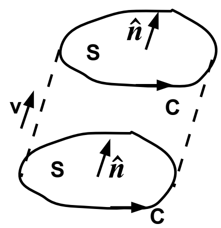

Figure 13: A moving circuit for Faraday's law.

To proceed we define the *electromotive force* as the circulation of the electric field

$$\mathcal{E} = \oint_C d\mathbf{r} \cdot \mathbf{E} \,. \tag{5.1}$$

Faraday's observation was that a time changing flux of the magnetic field F induces an electromotive force

$$\mathcal{E} = -\frac{1}{c}\frac{dF}{dt} \quad \Longrightarrow \quad \oint_C d\mathbf{r} \cdot \mathbf{E}' = -\frac{1}{c}\frac{d}{dt}\int_S dS\hat{\mathbf{n}} \cdot \mathbf{B} \,. \tag{5.2}$$

<u>**Remarks:**</u>

• The computation of the two integrals is done in the coordinate system in which the circuit is at rest. For that reason we denoted the electric field with a prime. We did not do the same for the magnetic field since the difference to leading order in the typical velocities of the sources as compared with the speed of light is considered negligible.

• The charge in the magnetic flux is due to the explicit dependence of $\mathbf{B}$ on time as well as due to the motion of the circuit which changes its location.

Then since the circuit is moving and using the convective derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ we have that

$$\begin{aligned} \frac{d}{dt}\int_S dS\hat{\mathbf{n}} \cdot \mathbf{B} &= \int_S dS\hat{\mathbf{n}} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{B}\right) , \\ &= \int_S dS\hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} + \oint_C d\mathbf{r} \cdot \mathbf{B} \times \mathbf{v} \,, \end{aligned} \tag{5.3}$$

where to obtain the second line I used the fact that there are no magnetic charge sources, that **v** is constant and Stoke's theorem. Hence (5.2) can be written as

$$\oint_C d\mathbf{r} \cdot \left(\mathbf{E}' - \frac{1}{c}\mathbf{v} \times \mathbf{B} \right) = -\frac{1}{c} \int_S dS \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} \, . \tag{5.4}$$

However the r.h.s. represents the charge of the magnetic flux as seen in a coordinate system fixed in the laboratory, i.e. that is our **E**. Hence

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B} \, . \tag{5.5}$$

This formula is valid for $v \ll c$, called the *non-relativistic limit*, but already it tells us that from the point of view of a moving observer w.r.t. a fixed laboratory, a magnetic field appears as if it is purely electric. The result is *Faraday's law*

$$\oint_C d\mathbf{r} \cdot \mathbf{E} = -\frac{1}{c} \int_S dS \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} \, . \tag{5.6}$$

Using Stoke's theorem and the fact that the surface S and the boundary C are arbitrary, we may turn the integral condition into the differential form

$$\boxed{\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}} \, , \tag{5.7}$$

which replaces the curl free condition in electrostatics (2.12).

5.1 Maxwell equations

Let's collect the system of equations for electricity and magnetism as we have developed them so far

$$\begin{aligned}
&\text{Gauss's Law :} && \nabla \cdot \mathbf{E} = 4\pi\rho \, , \\
&\text{Ampere's Law :} && \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \, , \\
&\text{Faraday's Law :} && \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \, , \\
&\text{No magnetic charges :} && \nabla \cdot \mathbf{B} = 0 \, .
\end{aligned} \tag{5.8}$$

It is an occasion in which mathematical consistency and beauty led to a physical law that is also consistent with experiment. The reason for the inconsistency is that from Ampere's law

$$0 = \nabla \cdot (\nabla \times \mathbf{B}) = \frac{4\pi}{c} \nabla \cdot \mathbf{J} = -\frac{4\pi}{c} \frac{\partial \rho}{\partial t} , \tag{5.9}$$

where we used the continuity equation (4.7). Hence consistency occurs only for time independent charge densities. The mathematical resolution is to perform the replacement $\mathbf{J} \to \mathbf{J} + \frac{1}{4\pi} \partial_t \mathbf{E}$. Then the r.h.s. of (5.9) acquires the extra term $\frac{1}{c} \partial_t (\nabla \cdot \mathbf{E})$ which upon using Coulomb's law produces the correct term to cancel the r.h.s. of (5.9).

This modification results into the celebrated Maxwell's equations for Electrodynamics

$$\boxed{\begin{array}{ll} \nabla \cdot \mathbf{E} = 4\pi\rho , & \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} , \\ \nabla \cdot \mathbf{B} = 0 , & \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} . \end{array}}$$

Before we study them further we mention that this system can be cast in a relativistically invariant form. In that sense the result (5.5) is a Galilean approximation to a relativistic transformation between the electric and magnetic fields in the coordinate system fixed on the moving circuit and that in the laboratory, which is valid for small velocities. Finally Maxwell's eqs. can be written very compactly using the mathematical language of differential forms, a convenient approach to calculus of many variables that is independent of coordinates. None of these formulations will be essential in this module, but you may encounter them in modules focused on Relativity and Manifolds.

5.1.1 Vector and scalar potentials

Recall the scalar and vector potentials introduced in (2.9) and (4.11). The latter is not modified but the first is. The result is

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} , \qquad \mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}} . \tag{5.10}$$

Substituting into Coulomb's law we obtain

$$\nabla^2 \Phi + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -4\pi\rho . \tag{5.11}$$

Similarly, substituting into the modified Ampere's eq. we obtain (**exercise**)

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{J}\,. \tag{5.12}$$

5.1.2 Gauge invariance revisited

These eqs. are coupled and we would like, if possible to decouple them. For that we utilize the gauge invariance transformation (4.28) we have briefly mentioned for magnetostatics. For the Maxwell equations we find that this transformation reads

$$\mathbf{A} \to \mathbf{A} + \nabla\Psi\,, \qquad \Phi \to \Phi - \frac{1}{c}\frac{\partial\Psi}{\partial t}\,, \tag{5.13}$$

since then the fields in (5.10) are left invariant. It is important to note that the function $\Psi(\mathbf{r}, t)$ is an arbitrary function of space and time. Hence, it can be used to fix an arbitrary expression of Φ and $\mathbf{A}$ and/or of their derivatives to an arbitrary value. I stress that one can change the gauge to simplify the computations but this always preserves the physical fields which are the electric and magnetic field.

However, certain choices have advantages.

Lorentz gauge: A particularly convenient choice is

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0\,. \tag{5.14}$$

The advantage of this choice is that in the system of eqs. (5.11) and (5.12), Φ and $\mathbf{A}$ decouple

$$\begin{aligned}\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{J}\,,\\ \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -4\pi\rho\,,\end{aligned} \tag{5.15}$$

which is the *wave eq.* with sources. The speed of propagation is c.

The Lorentz gauge choice does not completely fix our freedom to transform the potentials Φ and $\mathbf{A}$. Substituting (5.13) into (5.14) we find that any function satisfying the wave eq.

$$\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}\,, \tag{5.16}$$

preserves the choice (5.14). This is called *remaining gauge freedom*.

Coulomb gauge: This corresponds to the gauge choice

$$\nabla \cdot \mathbf{A} = 0 \,. \tag{5.17}$$

Then the system of eqs. (5.11) and (5.12) becomes

$$\begin{aligned} \nabla^2 \Phi &= -\frac{4\pi}{c}\rho \,, \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\nabla\frac{\partial \Phi}{\partial t} \,. \end{aligned} \tag{5.18}$$

We may solve the first eq. for Φ, then substitute into the second and solve for $\mathbf{A}$. In this case the gauge freedom analogous to (5.16) is

$$\nabla^2 \Psi = 0 \,. \tag{5.19}$$

An advantage of this Coulomb gauhe is that the eq. obeyed by the scalar potential is the same as in the case of electrostatics.

The fact that the potentials Φ and $\mathbf{A}$ obey different equations, depending on the gauge choice, does not affect the electric and magnetic fields as well as all physical quantities obtained from them.

5.1.3 Retardation effects

We have seen that in the Lorentz gauge all potentials obey the same type of equation, i.e. wave equation with sources. For the general theory and our intuition we expect that if an electric and/or a magnetic field is turned on in a finite region in space, then their effect as observed at a distance will be noticed at a later time. To see that consider the scalar equation (5.15) for a charge source which is well localized in space and time, i.e.

$$\rho(\mathbf{r}, t) = p\delta(t)\delta^{(3)}(\mathbf{r}) \,, \tag{5.20}$$

where p is a constant with units of charge $\times$ time. Then we have to solve the equation

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi p\delta(t)\delta^{(3)}(\mathbf{r}) \,, \tag{5.21}$$

in all 3-dim space and for all times. Fourier expanding in time

$$\Phi(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \Phi_\omega(\mathbf{r}) e^{-i\omega t} \quad \Longleftrightarrow \quad \Phi_\omega(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega \Phi(\mathbf{r},t) e^{i\omega t} \tag{5.22}$$

and using the last of (E.5) we obtain

$$\nabla^2 \Phi_\omega + k^2 \Phi_\omega = -4\pi p \delta^{(3)}(\mathbf{r}) \ , \tag{5.23}$$

which is the inhomogeneous *Helmholtz eq.* with radially symmetric solution

$$\Phi_\omega = \frac{1}{r}\left(C_+ e^{ikr} + C_- e^{-ikr}\right) \ , \tag{5.24}$$

where $A_\pm$ are constants of integration obeying $C_+ + C_- = 1$ (**exercise**). Then, after inverting the Fourier transform we find the solution as the sum of two terms involving delta-functions

$$\Phi = \frac{p}{r}\left[C_+ \delta\left(t - \frac{r}{c}\right) + C_- \delta\left(t + \frac{r}{c}\right)\right] \ , \tag{5.25}$$

where we have used the last of (E.5). Clearly, if we want *causality* that is that the result is observed after the event that caused it we have to choose $C_- = 0$ and $C_+ = 1$. Then the solution is

$$\Phi = \frac{p}{r}\delta\left(t - \frac{r}{c}\right) \ . \tag{5.26}$$

Note that if the propagation speed c was infinity, then the potential would have been felt at all distances r from the source at the same time no matter how far from it. Due to the finiteness of c the source sends a signal which travels as a wave with speed c and the wavefront is at $r = ct$.

The general conclusion one draws is that there is a *retardation time* until an observer realizes that a change in the charge and currents sources has occurred. This information travels with speed c. Retardation effects become important when the velocities and accelerations associated with the sources are high. If v is a typical velocity in the source distributions a necessary but maybe not sufficient conditions that these effects can be neglected is $v \ll c$.

5.2 Fields of moving charges

We would like to know what the electric and magnetic field of a moving charge are. When the charge is at rest then there is only electric field given by (2.4). When the charge is moving retardation effects have to be taken into account. In addition to that since the charge as it moves creates a current we expect a non-vanishing magnetic field.

Assume a charge q has a specified trajectory $\mathbf{r}_0(t)$. Then the corresponding charge and current densities are

$$\rho = q\delta^{(3)}(\mathbf{r} - \mathbf{r}_0(t)) , \qquad \mathbf{J} = q\mathbf{v}\delta^{(3)}(\mathbf{r} - \mathbf{r}_0(t)) . \tag{5.27}$$

where $\mathbf{v} = \frac{d\mathbf{r}_0(t)}{dt}$ is the corresponding velocity.[3] Let an observation be at a point $\mathbf{r}$. Due to retardation effects what we observed at time t is not what has happened at that time but at an earlier time t' given by

$$\boxed{t' = t - \frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}} . \tag{5.28}$$

This is algebraic equation that could be difficult to solve, depending on how complicated the r.h.s. is due to the trajectory $\mathbf{r}_0(t)$. In general t' is then a function of t and $\mathbf{r}$. A solution to this equation is not guaranteed to exist in general. For instance if a charge is created at $t = 0$ its presence cannot be felt by another charge far away immediately after. However, if a solution exists it is unique. To see that assume that there exist two solutions to (5.28), namely t'_1 and $t'_2 > t'_1$. Then by subtracting we find that

$$c(t'_2 - t'_1) = |\mathbf{r}-\mathbf{r}_0(t'_1)| - |\mathbf{r}-\mathbf{r}_0(t'_2)| \leqslant |\mathbf{r}_0(t'_2)-\mathbf{r}_0(t'_1)| , \tag{5.29}$$

by the triangle inequality. Unless $t'_2 = t'_1$, this then implies that the average velocity of the charge between t'_1 and t'_2 is larger than the speed of light c, which is impossible.

For convenience we define the velocity relative to c, the distance R and the unit vector $\hat{\mathbf{n}}$ from the particle to the observation point as

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c} , \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} , \qquad R = |\mathbf{r}-\mathbf{r}_0(t)| , \qquad \hat{\mathbf{n}} = \frac{\mathbf{r}-\mathbf{r}_0(t)}{R} . \tag{5.30}$$

[3] One may show that these obey the continuity eq. (4.7) (**exercise**).

Then in the Lorentz gauge the scalar and vector potentials are given by

$$\boxed{\Phi(\mathbf{r},t) = \frac{q}{(1-\boldsymbol{\beta}\cdot\hat{n})R}\Big|_{\rm ret}\ , \qquad \mathbf{A}(\mathbf{r},t) = \frac{q\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\hat{n})R}\Big|_{\rm ret}}\ . \tag{5.31}$$

where the subscript emphasizes that the time in which the r.h.s. is evaluated is at the retarded time $t' = t'(\mathbf{r},t)$. These are called the *Lienard–Wiechert potentials*. In the limit of $\boldsymbol{\beta} \ll 1$, $\mathbf{A} = 0$ and Φ becomes the same as in electrostatic case (2.11). We will not give here the proof of (5.31) which can be found or sketched in standard textbooks. Plugging next these expressions into (5.10) we may obtain the electric and magnetic fields produced by a moving particle. The derivation can be done by using various methods. In the most straightforward and pedagogical, nevertheless not the shortest way, one simply computes all derivatives keeping in mind that the retardation time t' depends on the time as well as the observation point, i.e. $t' = t'(\mathbf{r},t)$. The end result for the electric field is

$$\boxed{\mathbf{E}(\mathbf{r},t) = \frac{q}{\gamma^2 R^2(1-\boldsymbol{\beta}\cdot\hat{n})^3}(\hat{\mathbf{n}}-\boldsymbol{\beta}) + \frac{q}{cR(1-\boldsymbol{\beta}\cdot\hat{n})^3}\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\Big|_{\rm ret}}\ . \tag{5.32}$$

For the magnetic field one finds that

$$\boxed{\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{n}}\times\mathbf{E}\big|_{\rm ret}}\ . \tag{5.33}$$

<u>**Remarks:**</u>

• The electric field has two terms. The first term depends only on the velocity and falls off as $1/r^2$ for large distances. The second term vanishes for zero acceleration and falls off as $1/r$ for large distances. This behavior is responsible for the radiation emitted by accelerated particles as we shall see.

• The magnetic and electric fields as perpendicular.

<u>**Exercise:**</u> To get an idea of how (5.32) can be derived using (5.10), use that

$$\partial_t t' = 1 - \frac{1}{c}\partial_t R\ , \qquad \partial_i t' = -\frac{1}{c}\partial_i R \tag{5.34}$$

to derive

$$\partial_i R_j = \delta_{ij} - \partial_i t' v_j = \delta_{ij} + \beta_j \partial_i R\ . \tag{5.35}$$

After some algebraic manipulations involving for instance multiplying the above by R_j and summing over j, show that

$$\partial_i R_j = \delta_{ij} + \frac{\hat{n}_i \beta_j}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} , \qquad \partial_t R_i = -\frac{v_i}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} ,$$
$$\partial_i R = \frac{\hat{\mathbf{n}}}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} , \qquad \partial_t R = -\frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} , \qquad \frac{dt}{dt'} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}} . \qquad (5.36)$$

5.2.1 Fields of a moving charge with constant velocity

If the velocity is constant only the first term in (5.32) contributes. Let's set up our coordinate system as in the Fig. 14 so that the observation point is at $(0, b, 0)$ and the particle moves with velocity $(v, 0, 0)$. In addition we measure time so that at $t = 0$ the particle is at the origin of our coordinate system.

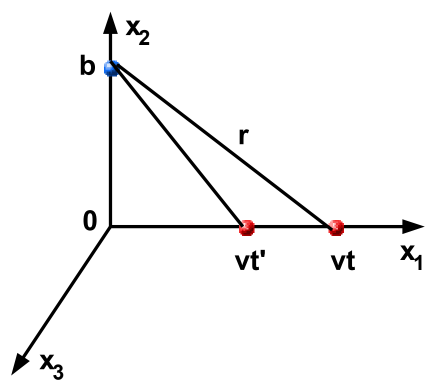

Figure 14: Set up for computing the electric and magnetic fields of a moving electric charge with constant velocity.

We have that

$$R(t) = \sqrt{b^2 + c^2 t^2} , \qquad \hat{n} = \frac{1}{R(t)} (-vt, b, 0) . \qquad (5.37)$$

The retarded time is computed from (5.28) which in our case becomes

$$t' = t - \frac{1}{c} \sqrt{b^2 + c^2 t'^2} . \qquad (5.38)$$

The unique solution is

$$t' = \gamma^2 t - \gamma \sqrt{\beta^2 \gamma^2 t^2 + b^2 / c^2} . \qquad (5.39)$$

Then using the above we compute that (**Exercise**)

$$\hat{\mathbf{n}} = \frac{1}{t-t'}(-\beta t', b/c, 0)\,, \qquad \hat{\mathbf{n}} - \boldsymbol{b} = \frac{1}{t-t'}(-\beta t, b/c, 0)\,.$$

$$\hat{\mathbf{n}} \cdot \mathbf{fi} = \beta^2 \frac{t'}{t-t'}\,, \qquad R^2 = c^2(t-t')^2\,. \tag{5.40}$$

After a little algebra

$$\boxed{\mathbf{E} = \frac{q\gamma}{(b^2+\gamma v^2 t^2)^{3/2}}(-vt, b, 0)\,, \qquad \mathbf{B} = \frac{q\gamma}{(b^2+\gamma v^2 t^2)^{3/2}}(0, 0, b\beta)}\,. \tag{5.41}$$

The magnetic and electric fields are obviously orthogonal.

Evidently, as β approaches unity, the electric field lines become denser in the direction perpendicular to the motion of the particle relatively to the direction along its motion. This is seen from the functional form of the fields as depicted in Fig. 15.

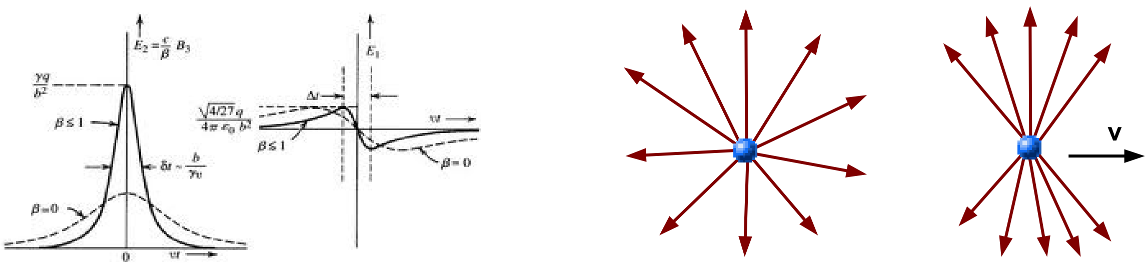

Figure 15: a) Electric field components of a moving electric charge based on (5.41). b) Electric field lines for a positive charge at rest and when it moves horizontally.

In the extreme limit $\beta \to 1$ the component E_2 approaches a delta function, i.e.

$$E_2 = \lim_{\gamma\to\infty} \frac{qb\gamma}{(b^2+\gamma^2 c^2 t^2)^{3/2}} = \lambda\delta(t)\,. \tag{5.42}$$

The proportionality constant is found by integrating both sides

$$\lambda = \frac{q}{bc} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{3/2}}}_{2} = 2\frac{q}{bc}\,, \tag{5.43}$$

where in order to integrate I used the variable $z = c\gamma t/b$. Also $E_1 \sim t\delta(t) = 0$. Hence in the limit of ultrahigh velocities, i.e. $v \to c$, equivalently $\beta \to 1$

$$E_2 = B_3 = \frac{2q}{bc}\delta(t) \qquad (5.44)$$

and all other components become zero. Hence such a charged particle produces a *shock wave* type of field. It is everywhere zero except at the plane perpendicular to the position of the particle where however it is extremely strong.

Exercise: It we would be nice to present our previous results for the fields of charged particle moving with constant velocity, using a specific coordinate system in particular polar cylindrical coordinates. Let the particle move with $\mathbf{v} = v\hat{z}$. Then show that the potentials

$$\Phi = \frac{q\gamma}{\sqrt{\rho^2 + (z - \beta ct)^2\gamma^2}} , \qquad \mathbf{A} = \frac{q\gamma\beta\hat{z}}{\sqrt{\rho^2 + (z - \beta ct)^2\gamma^2}} , \qquad (5.45)$$

in polar coordinates, solve (5.15). Then using (5.10) compute the electric and magnetic fields and show that your expressions when they are specified to the coordinate system of Fig. (14) agree with (5.41). Show also the electric field can be rewritten as

$$\mathbf{E} = \frac{q}{\gamma^2}\frac{1}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{\hat{R}}{R^2} , \qquad \mathbf{R} = \mathbf{r} - \mathbf{v}t , \qquad (5.46)$$

where we refer to the Fig. 16.

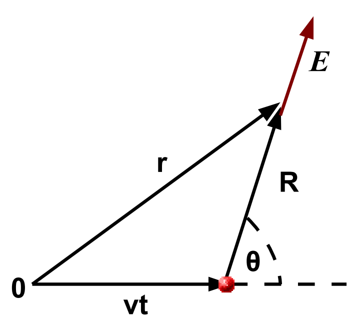

Figure 16: Electric field of a moving electric charge.

6 Radiation

It is a fact that charges when they accelerate loose energy. Hence, in order to maintain their motion, e.g. circular, we have to continuously supply them with energy. This has important consequences both theoretical and experimental. For instance, in big particle accelerators, such as the one at CERN, supplying enough energy to the accelerated charged particles is a major issue when their velocities reach values close to the speed of light. This loss of energy by accelerated charged particles is unlike what you might be familiar with from classical mechanics.

6.1 Energy flow in electromagnetic fields

Consider a single charge q moving in a region where there are electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ with velocity $\mathbf{v}$. The rate of doing work is $q\mathbf{v}\cdot\mathbf{E}$. In the presence of currents this becomes $\int_V dv \mathbf{J}\cdot\mathbf{E}$. Then from Ampere's law

$$\int_V dv \mathbf{J}\cdot\mathbf{E} = \frac{1}{4\pi}\int_V dv \left[c\, \overbrace{\mathbf{E}\cdot(\nabla\times\mathbf{B})}^{\mathbf{B}\cdot\nabla\times\mathbf{E}-\nabla\cdot(\mathbf{E}\times\mathbf{B})} - \mathbf{E}\cdot\partial_t\mathbf{B} \right)$$

$$= -\frac{1}{4\pi}\int_V dv \left(c\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \frac{1}{2}\partial_t E^2 + \frac{1}{2}\partial_t B^2 \right) , \qquad (6.1)$$

where I used Faraday's law. Realizing that

$$\boxed{U = \frac{1}{8\pi}(E^2+B^2)} , \qquad (6.2)$$

is the energy density of the electromagnetic field we may write the above as

$$\int_V dv \left[\frac{\partial U}{\partial t} + \frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \mathbf{J}\cdot\mathbf{E} \right] = 0 . \qquad (6.3)$$

Since the volume V is arbitrary we obtain

$$\boxed{\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}} , \qquad (6.4)$$

where

$$\boxed{\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}} , \qquad (6.5)$$

is called the *Poynting vector*. The interpretation of (6.4) is similar to that of the continuity eq. (4.7). The time change of the energy density of the electromagnetic field within a certain volume plus the energy flowing out of the surface equals the negative work done on the sources by the electromagnetic field. Hence, the Poynting vector represents the energy flowing per unit area and per unit time, see Fig. 17.

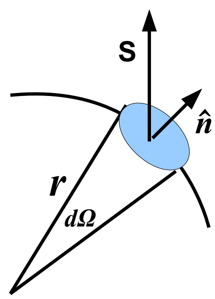

Figure 17: Energy flow and the Poynting vector.

The total power, i.e. energy per unit time emitted per solid angle $d\Omega$ is

$$\frac{dP(t)}{d\Omega} = r^2 \mathbf{S}\cdot\hat{\mathbf{n}}\Big|_{\rm ret} = \frac{c}{4\pi} r^2 (\mathbf{E}\times\mathbf{B})\cdot\hat{\mathbf{n}}\Big|_{\rm ret} . \tag{6.6}$$

The energy emitted per solid angle is then

$$\frac{dW}{d\Omega} = \int dt \frac{dP(t)}{d\Omega} . \tag{6.7}$$

Since the integrand depends on the retarded time t' it is convenient to turn the integration into an integration over t'. This is done by noting that $dt = \frac{dt}{dt'}dt'$ and then using the appropriate expression in (5.36)

$$\frac{dW}{d\Omega} = \int dt' \frac{d\tilde{P}(t')}{d\Omega} , \tag{6.8}$$

where

$$\frac{d\tilde{P}(t')}{d\Omega} = (1 - \boldsymbol{\beta}\cdot\hat{\mathbf{n}}) \frac{dP(t)}{d\Omega}\Big|_{\rm ret} . \tag{6.9}$$

In addition

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times(\hat{\mathbf{n}}\times\mathbf{E}) = \frac{c}{4\pi}(E^2\hat{\mathbf{n}} - (\mathbf{E}\cdot\hat{\mathbf{n}})\mathbf{E}) , \tag{6.10}$$

from which

$$\mathbf{S}\cdot\mathbf{n} = \frac{c}{4\pi}(E^2 - (\mathbf{E}\cdot\hat{\mathbf{n}})^2) = \frac{c}{4\pi}|\hat{\mathbf{n}}\times\mathbf{E}|^2 . \tag{6.11}$$

Finally, we will assume that we observe the radiation at a distance r much larger than the typical distance traveled during the acceleration of the particle, i.e. $r \gg |\mathbf{r}_0(t)|$ for the time that the particle is accelerated. Then $R \simeq r$ and also we keep in (5.32) only the second term involving the acceleration. Then[4]

$$\boxed{\frac{d\tilde{P}(t')}{d\Omega} = \frac{q^2}{4\pi c}\frac{|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^5}} . \tag{6.12}$$

A particular case of interest is when the velocity is periodic in time with period T. Evidently, the total radiated power diverges. What makes more sense is to compute the average power per solid angle radiated over an entire period. This is defined as

$$\frac{d\bar{P}}{d\Omega} = \frac{1}{T}\int_0^T dt' \frac{d\tilde{P}(t')}{d\Omega} . \tag{6.13}$$

The total power radiated in a single period is obtained by integrating over the solid angle.

The above general formulae will be applied in two cases of particular interest.

6.2 Linear accelerated motion: $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$

We consider first the case in which the velocity and acceleration are collinear, i.e. rectilinear motion. Let the angle between $\mathbf{v}$ and the direction to the observation point $\hat{\mathbf{n}}$ be θ. From (6.12) we compute that

$$\frac{d\tilde{P}_{\parallel}(t)}{d\Omega} = \frac{q^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} . \tag{6.14}$$

The maximum power is emitted at an angle $\theta_{\max}$ found from

$$\partial_\theta\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right) = 0 . \tag{6.15}$$

We find that

$$\cos\theta_{\max} = \frac{\sqrt{15\beta^2+1}-1}{3\beta} , \tag{6.16}$$

where we kept only one solution of the quadratic eq. in order to have $|\cos\theta_{\max}| \leqslant 1$.

[4]Note the order of performing the exterior product which is not an associative operation.

For $\beta \ll 1$ one finds that $\theta_{\text{max}} \simeq \dfrac{\pi}{2} - \dfrac{5\beta}{2}$. In the opposite limit $\gamma \gg 1$ we have $\theta_{\text{max}} \simeq \frac{1}{2\gamma}$. In conclusion, for small velocities most of the radiation is emitted in a small cone in the direction perpendicular to the motion. For velocities near the speed of light most of the radiation is emitted in a narrow cone along the direction of the particle. Graphical details are depicted in Fig. 18.

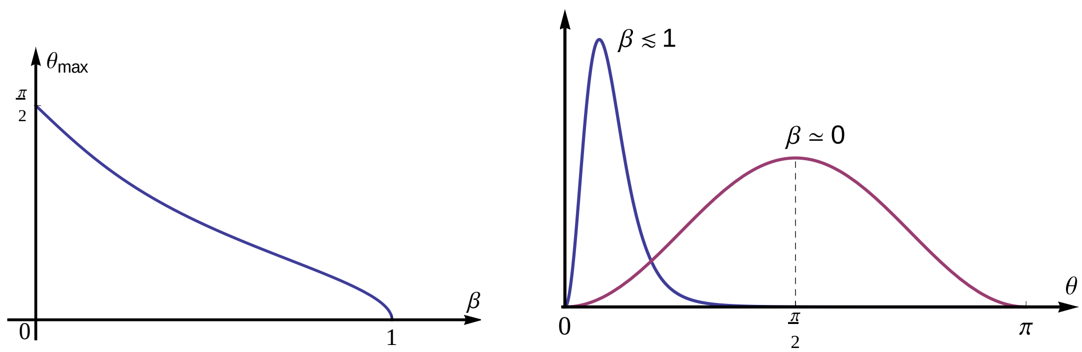

Figure 18: a)The angle where maximum radiation is emitted as a function of β. b)Power emitted per solid angle based on (6.14).

The total power is

$$\tilde{P}_{\|}(t') = \int d\Omega \frac{d\tilde{P}(t')}{d\Omega} = \frac{q^2\dot{v}^2}{4\pi c^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$
$$= \frac{q^2\dot{v}^2}{2c^3}\underbrace{\int_{-1}^{1} dx \frac{1-x^2}{(1-\beta x)^2}}_{\frac{4}{3}(1-\beta^2)^{-3}}, \qquad (6.17)$$

giving

$$\tilde{P}_{\|}(t) = \frac{2}{3}\frac{q^2|\dot{\mathbf{v}}|^2}{c^3}\gamma^6 . \qquad (6.18)$$

<u>**Exercise:**</u> Work out the details of this subsection.

6.3 Instantaneous circular motion: $\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$

The next interesting motion we consider is that of a charged particle performing a motion in which for some amount of time the velocity and acceleration are perpendicular. We set up a coordinate system as in the Fig. 19.

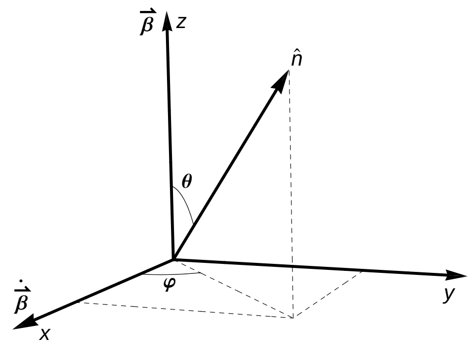

Figure 19: Coordinate system choice for an instantaneous circular motion.

We have

$$\boldsymbol{\beta} = \beta(0,0,1)\ , \qquad \dot{\boldsymbol{\beta}} = |\dot{\boldsymbol{\beta}}|(1,0,0)\ , \qquad \hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\ . \tag{6.19}$$

In order to use our general eq. (6.12) we compute

$$(\hat{\mathbf{n}}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} = (\cos\theta - \beta)|\dot{\boldsymbol{\beta}}|\hat{y} - \sin\theta\sin\phi|\dot{\boldsymbol{\beta}}|\hat{z} \tag{6.20}$$

and

$$\begin{aligned}\hat{\mathbf{n}} \times [(\hat{\mathbf{n}}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] = |\dot{\boldsymbol{\beta}}|\big[\cos\theta(\beta - \cos\theta)\\ - \sin^2\theta\sin^2\phi\hat{x} + \sin^2\theta\sin\phi\cos\phi\hat{y} + (\beta - \cos\theta)\sin\theta\cos\phi\hat{z}\big]\ .\end{aligned} \tag{6.21}$$

Then

$$|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]^2 = |\dot{\boldsymbol{\beta}}|^2 \left[(1 - \beta\cos\theta)^2 - \gamma^{-2}\sin^2\theta\cos^2\phi\right]\ . \tag{6.22}$$

Therefore the power emitted be solid angle is

$$\frac{d\tilde{P}_\perp}{d\Omega} = \frac{q^2|\dot{\mathbf{v}}|^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]\ . \tag{6.23}$$

The total power is obtained by integration over all angles

$$\tilde{P}_\perp(t') = \frac{q^2|\dot{\mathbf{v}}|^2}{4\pi c^3}\Big[\int_0^{2\pi} d\phi \underbrace{\int_0^{\pi} d\theta \frac{\sin\theta}{(1-\beta\cos\theta)^3}}_{2(1-\beta^2)^{-3}}$$
$$-\gamma^{-2}\underbrace{\int_0^{2\pi} d\phi \cos^2\phi}_{\pi} \underbrace{\int_0^{\pi} d\theta \frac{\sin^3\theta}{(1-\beta\cos\theta)^5}}_{\frac{4}{3}(1-\beta^2)^{-3}}\Big] . \quad (6.24)$$

After some algebra we obtain

$$\tilde{P}_\perp(t') = \frac{2}{3}\frac{q^2|\dot{\mathbf{v}}|^2}{c^3}\gamma^4 . \quad (6.25)$$

Note that the similarity of this expression to the one in (6.18). The only difference is the power of γ. For small velocities both expressions reduce to

$$\boxed{P_{\text{Larmor}}(t) = \frac{2}{3}\frac{q^2|\dot{\mathbf{v}}|^2}{c^3}} , \quad (6.26)$$

where we dropped the tilde since there is no distinction between the retardation time t' and t. This is the Larmor formula giving the total power radiated by a nonrelativistic point charge as it accelerates or decelerates. This formula is very general and applies to all kinds of motion, not just rectilinear or instantaneously circular.

Exercise: Work out the details of this subsection.

Remark: Larmor's formula breaks down not only when the velocities are no longer small when compared with the speed of light, but also at distances comparable or smaller to the atomic scale where quantum mechanics dominates. In particular, let's think of the Hydrogen atom, as the classical system of one proton and one electron. Then since the electron would necessarily be loosing energy due to the fact that it is accelarating around the proton, the atom should eventually collapse. In fact a relatively simple computation shows that if the hypothesis for validity of the classical description of the atom holds, this should happen in approximately $10^{-11}sec$! The fact that this is not happening tells us with no ambiguity that Classical Electrodymanics is not describing Nature at the atomic scale.

6.4 Energy loss in linear vs circular colliders

One naturally wonders when is a charged particle loosing energy faster? When it accelerates in a linear motion or in a circular? If we naively compare (6.18) and (6.25) it appears as if more energy is lost during linear acceleration. However, this is not correct since one has to express $|\dot{\mathbf{v}}|$ in terms of the force or energy supplied by a machine.

For this subsection elements of the Special Theory of Relativity will be needed. All necessary formulae will be provided, however, without proof or further discussion.

Rectilinear motion: In that case the momentum is $p = mc\beta\gamma$ and we would like to compute its time derivative. From the identity

$$\beta^2\gamma^2 + 1 = \frac{1}{1-\beta^2} \implies 2(\beta\gamma)\frac{d(\beta\gamma)}{dt} = 2\beta\gamma^4\frac{d\beta}{dt}\ , \tag{6.27}$$

we obtain that

$$\dot{\beta} = \frac{\dot{p}}{mc\gamma^3}\ . \tag{6.28}$$

However, the change in momenta is associated with the force needed to achieve that, or with the amount of energy one should supply per unit length. Then (6.18) can be written as

$$\tilde{P}_{\parallel} = \frac{2}{3}\frac{q^2}{m^2c^3}\dot{p}^2 = \frac{2}{3}\frac{q^2}{m^2c^3}(\partial_x W)^2\ . \tag{6.29}$$

Circular motion: In that case the magnitude of the velocity does not change, so that γ is constant. Therefore from $\mathbf{p} = mc\gamma\boldsymbol{\beta}$ and immediately we compute that

$$\tilde{P}_{\perp} = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2|\dot{\mathbf{p}}|^2 = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2(\partial_x W)^2\ . \tag{6.30}$$

Remarks;

• In both cases, the heavier the particle the smaller the energy loss. Energy loss is maximized for electrons.

• For the same force, i.e. same $\partial_x W$, the energy loss is larger in circular accelerators

$$\frac{\tilde{P}_{\perp}}{\tilde{P}_{\parallel}} = \gamma^2\ . \tag{6.31}$$

For velocities near the speed of light, this ratio is very large. For example for $\beta =$

0.9999 we have $\gamma^2 = 5000$. Hence for the same velocity it seems to cost much more to maintain the velocity in a circular accelerator than in a linear one. However, in order to achieve these velocities in a linear accelerator we would need a distance which is of the order of many Km and no technology can provide strong enough electric fields along such long paths. On the other hand in a circular accelerator we may reuse the same space again since the particles are looping.

Exercise: Consider a charged particle in simple harmonic motion in the z-axis with $z(t) = a\cos\omega t$, where a is the amplitude and the angular frequency is $\omega = \frac{2\pi}{T}$. Show that:

a) The average power per unit angle over a period is given by

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2 c\beta^4}{32\pi a^2}\frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}}\sin^2\theta\,, \tag{6.32}$$

where $\beta = a\omega/c$ and θ is the polar angle in spherical coordinates.

b) There is a critical values $\beta_{\text{crit}} = \frac{2}{\sqrt{15}} \simeq 0.52$ below which the maximum of the radiated power is at $\theta = \frac{\pi}{2}$, whereas for $\beta_{\text{crit}} \leqslant \beta < 1$ the maximum radiation is emitted at two angles which are symmetric around $\theta = \frac{\pi}{2}$.

c) Show that the total average radiated power is

$$\bar{P} = \frac{q^2 c}{12a^2}\beta^4\gamma^3(4-3\beta^2) \tag{6.33}$$

and that this is a monotonously increasing function of β.

A Kronecker delta and Levi–Civita symbols

The Kronecker delta symbol is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j\ , \\ 0 & \text{if } i \neq j\ , \end{cases} \qquad i,j = 1,2,\dots\ . \tag{A.1}$$

Let i, j, k take the values $1, 2$ and 3. The Levi–Civita symbol ϵ_{ijk} is $+1(-1)$ if i, j, k is an even (odd) permutation of $(i, j, k) = (1, 2, 3)$ and zero if any two indices are equal. Hence ϵ_{ijk} is antisymmetric in all three indices.

The following properties are particularly useful

$$\begin{aligned} &\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\ , \\ &\epsilon_{imn}\epsilon_{mnj} = 2\delta_{ij}\ , \\ &\epsilon_{ijk}\epsilon_{ijk} = 6\ . \end{aligned} \tag{A.2}$$

B Elements of differential calculus

Let a Cartesian system of coordinates (x, y, z) as in the Fig. 20.

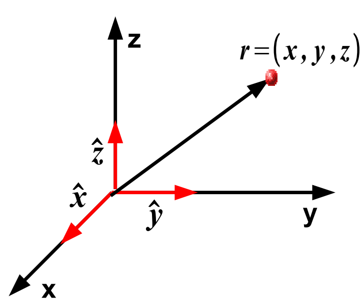

Figure 20: Cartesian system of coordinates with indicated unit vectors.

The position of a point in $\mathbb{R}^3$ with these coordinates will be denoted as

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}\ , \tag{B.1}$$

where $(\hat{x}, \hat{y}, \hat{z})$ are the corresponding orthogonal unit vectors.

The volume element in Cartesian coordinates is

$$dv = d^3\mathbf{r} = dxdydz \,. \tag{B.2}$$

The *gradient operator* is defined by its action on a function $\Phi(\mathbf{r})$ as

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{x} + \frac{\partial\Phi}{\partial y}\hat{y} + \frac{\partial\Phi}{\partial z}\hat{z} \,. \tag{B.3}$$

If we move the point at $\mathbf{x}$ by a small vector to a new position $\mathbf{r} + d\mathbf{r}$ the function changes as

$$d\Phi = \nabla\Phi \cdot d\mathbf{r} \,. \tag{B.4}$$

In the following $\mathbf{A}$ and $\mathbf{B}$ are vectors. In addition, it will be convenient to denote the Cartesian coordinates x, y, z by x_i, $i = 1, 2, 3$. Accordingly the components of a vector A_x, A_y and A_z will be denoted by A_1, A_2 and A_2, respectively.

The *divergence* of a vector is given by

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \partial_i A_i \,. \tag{B.5}$$

The *curl* of a vector is given by

$$\begin{aligned} \nabla \times \mathbf{A} &= \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_x \end{pmatrix} \,. \\ &= (\partial_y A_z - \partial_z A_y)\hat{\mathbf{x}} + (\partial_z A_x - \partial_x A_z)\hat{\mathbf{y}} + (\partial_x A_y - \partial_y A_x)\hat{\mathbf{z}} \,. \end{aligned} \tag{B.6}$$

The *Laplace operator* on a function $\Phi(\mathbf{x})$ acts as

$$\nabla^2\Phi = \nabla \cdot \nabla\Phi = \partial_i\partial_i\Phi \,. \tag{B.7}$$

An arbitrary *unit vector* is of the form

$$\hat{\mathbf{n}} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z} \,, \qquad n_x^2 + n_y^2 + n_z^2 = 1 \,. \tag{B.8}$$

Then the directional derivative along $\hat{n}$ is defined as

$$\frac{\partial}{\partial n} = \hat{\mathbf{n}} \cdot \nabla \; . \tag{B.9}$$

The ith component of the outer product $\mathbf{A} \times \mathbf{B}$ is given by

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k \; , \tag{B.10}$$

for any two vectors in $\mathbb{R}^3$.

<u>Exercise</u>: Using properties of determinants prove that the definition

$$\mathbf{A} \times \mathbf{B} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \; , \tag{B.11}$$

is consistent with (B.10).

<u>Exercise</u>: Using (A.2) show (some of) the following identities

$$\begin{aligned}
\nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \; , \\
\nabla(\Phi \mathbf{A}) &= \Phi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \Phi \; , \\
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \; , \\
\nabla \times (\Phi \mathbf{A}) &= \Phi(\nabla \times \mathbf{A}) + \nabla \Phi \times \mathbf{A} \; , \\
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \; , \qquad \text{(B.12)} \\
\nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \; , \\
\nabla \times (\nabla \Phi) &= 0 \; , \\
\nabla \cdot (\nabla \times \mathbf{A}) &= 0 \; .
\end{aligned}$$

<u>Exercise</u>: Show that

$$\begin{aligned}
&\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \; , \\
&(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})
\end{aligned} \tag{B.13}$$

and that the exterior product is non-associative, but the Jacobi identity is obeyed

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0\,. \tag{B.14}$$

C Fundamental theorems

C.1 The fundamental theorem for gradients

Consider the infinitesimal change (B.4) along a curve from the point **a** to the point **b**

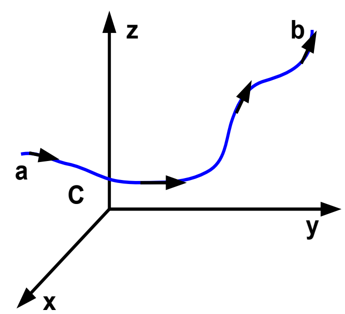

Figure 21: A curve C with end points **a** and **b**. Arrows indicate small tangent displacements.

The total change in Φ is

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla\Phi\cdot d\mathbf{r}=\Phi(\mathbf{b})-\Phi(\mathbf{a})\,. \tag{C.1}$$

This states that the total change of a function depends on the difference of values of the function at the end points and not on the details of the path between them. If the path is closed then the integral vanishes.

C.2 The fundamental theorem for divergences

Consider a volume V bounded by a surface S. The divergence (or Gauss's) theorem states that

$$\int_V dv\nabla\cdot\mathbf{A}=\oint_S dS\hat{\mathbf{n}}\cdot\mathbf{A}\,, \tag{C.2}$$

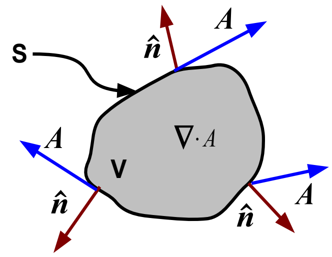

Figure 22: The divergence of a vector **A** over a volume V can be computed from its projection on the normal $\hat{\mathbf{n}}$ (pointing outwards) to the surface S bounding V.

C.3 The fundamental theorem for curls

Consider a surface S bounded by a closed curve C. The curl (or Stoke's) theorem states that

$$\int dS\hat{\mathbf{n}} \cdot (\nabla \times \mathbf{A}) = \oint_C \mathbf{A} \cdot d\mathbf{r} \,. \tag{C.3}$$

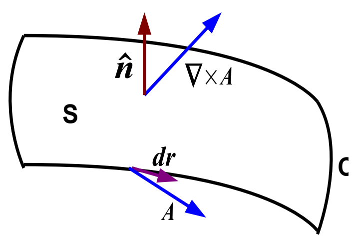

Figure 23: The projection of the curl of a vector **A** on the normal $\hat{\mathbf{n}}$ to a surface S can be computed from the circulation of **A** on the curve C (anticlockwise) bounding S.

D Curvilinear coordinates

In many problems there are present special symmetries we would like to take advantage in trying to solve them. Hence, using Cartesian coordinates is not always the most convenient choice. The most common other coordinates systems are the so called cylindrical and spherical.

D.1 Cylindrical coordinates

The change of variables from Cartesian coordinates is

$$\begin{aligned} & x = \rho\cos\phi\,, \qquad y = \rho\sin\phi\,, \qquad z = z\,, \\ & \rho \geqslant 0\,, \qquad 0 \leqslant \phi \leqslant 2\pi\,, \qquad -\infty < z < \infty\,. \end{aligned} \tag{D.1}$$

The relation between the orthogonal unit vectors is

$$\begin{aligned} \hat\rho &= \cos\phi\,\hat x + \sin\phi\,\hat y\,, \\ \hat\phi &= -\sin\phi\,\hat x + \cos\phi\,\hat y\,, \\ \hat z &= \hat z\,. \end{aligned} \tag{D.2}$$

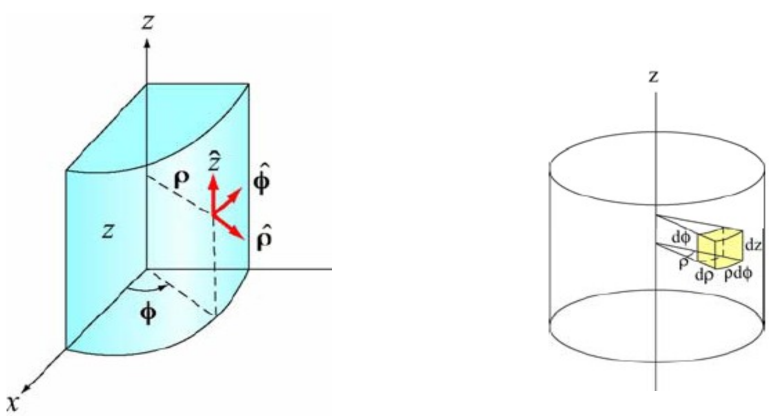

Figure 24: Cylindrical coordinates and the corresponding volume element.

Then for a function $\Phi = \Phi(\rho,\phi,z)$ and a vector $\mathbf{A} = A_\rho\hat\rho + A_\phi\hat\phi + A_z\hat z$ we have the following operations

$$\begin{aligned} \nabla\Phi &= \partial_\rho\Phi\hat\rho + \frac{1}{\rho}\partial_\phi\Phi\hat\phi + \partial_z\Phi\hat z\,, \\ \nabla\cdot\mathbf{A} &= \frac{1}{\rho}\partial_\rho(\rho A_\rho) + \frac{1}{\rho}\partial_\phi A_\phi + \partial_z A_z\,, \\ \nabla\times\mathbf{A} &= \frac{1}{\rho}(\partial_\phi A_z - \rho\partial_z A_\phi)\hat\rho + (\partial_z A_\rho - \partial_\rho A_z)\hat\phi + \frac{1}{\rho}\left[\partial_\rho(\rho A_\phi) - \partial_\phi A_\rho\right]\hat z\,, \\ \nabla^2\Phi &= \frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Phi) + \frac{1}{\rho^2}\partial^2_\phi\Phi + \partial^2_z\Phi\,. \end{aligned} \tag{D.3}$$

By replacing in (D.2) the unit vectors by vector components, $\hat\rho \to A_\rho$, etc and $\hat x \to A_x$, etc, we get the relation between components of vectors in the Cartesian and the polar coordinate systems.

The volume element in polar coordinates is

$$dv = d^3\mathbf{r} = \rho d\rho dz d\phi \ . \tag{D.4}$$

In cylindrical coordinates the normal unit vector on the curved surface of a cylinder is $\hat{\mathbf{n}} = \hat{\rho}$, whereas on the upper (lower) cap is $\hat{\mathbf{n}} = \hat{z}$ ($\hat{\mathbf{n}} = -\hat{z}$).

D.2 Spherical coordinates

The change of variables from Cartesian coordinates is

$$\begin{aligned} &x = r\sin\theta\cos\phi\ , \qquad y = r\sin\theta\sin\phi\ , \qquad z = r\cos\theta\ , \\ &r \geqslant 0\ , \qquad 0 \leqslant \theta \leqslant \pi\ , \qquad 0 \leqslant \phi \leqslant 2\pi\ . \end{aligned} \tag{D.5}$$

The relation between the orhtogonal unit vectors is

$$\begin{aligned} \hat{r} &= \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}\ , \\ \hat{\theta} &= \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}\ , \\ \hat{\phi} &= -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}\ . \end{aligned} \tag{D.6}$$

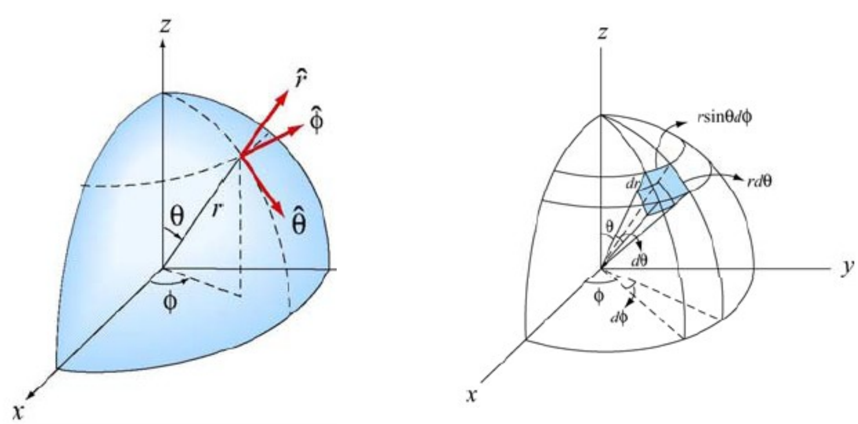

Figure 25: Spherical coordinates and the corresponding volume element.

Then for a function $\Phi = \Phi(r, \theta, \phi)$ and a vector $\mathbf{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$ we have

$$\begin{aligned}
\nabla\Phi &= \partial_r\Phi\hat{r} + \frac{1}{r}\partial_\theta\Phi\hat{\theta} + \frac{1}{r\sin\theta}\partial_\phi\Phi\hat{\phi}\ , \\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\partial_r(r^2A_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\partial_\phi A_\phi\ , \\
\nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta A_\phi) - \partial_\phi A_\theta\right]\hat{r} + \frac{1}{r\sin\theta}\left[\partial_\phi A_r - \sin\theta\partial_r(rA_\phi)\right]\hat{\theta} \\
&\quad + \frac{1}{r}\left[\partial_r(rA_\theta) - \partial_\theta A_r\right]\hat{\phi}\ , \qquad \text{(D.7)} \\
\nabla^2\Phi &= \frac{1}{r^2}\partial_r(r^2\partial_r\Phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\Phi\ .
\end{aligned}$$

By replacing the unit vectors in (D.6) by vector components, $\hat{r} \to A_r$, etc and $\hat{x} \to A_x$, we get the relation between components of vectors.

The volume element in polar coordinates is

$$dv = d^3\mathbf{r} = r^2 dr d\Omega\ , \qquad d\Omega = \sin\theta d\theta d\phi\ . \qquad \text{(D.8)}$$

The element $d\Omega$ is the elementary surface of the unit sphere, i.e. the sphere of radius. The finite element is called *solid angle* and is depicted in Fig. 25. Also note that

$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi\ . \qquad \text{(D.9)}$$

In spherical coordinates the normal unit vector on the surface of a sphere is $\hat{\mathbf{n}} = \hat{r}$.

E Delta function $\delta(x)$

A very useful concept is that of the delta–function (or Dirac's Delta) which is used to describe point-like charges, or charges localized at two-dimensional surfaces or one-dimensional lines and similarly for currents.

One natural way to intoduce it, is to consider the function

$$P_\sigma(x) = \frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}}\ , \qquad \sigma > 0\ , \qquad \text{(E.1)}$$

which is such that

$$\int_{-\infty}^{\infty} dx P_\sigma(x) = 1 \,, \qquad \forall\, \sigma > 0 \,. \tag{E.2}$$

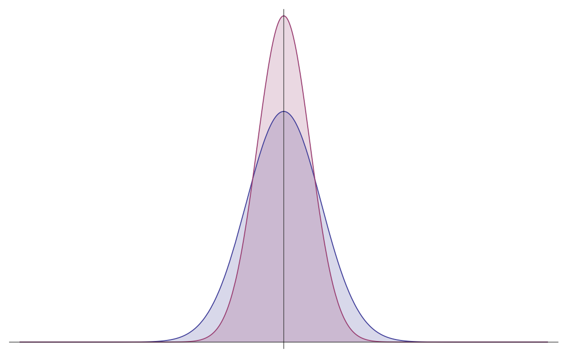

Figure 26: $P_\sigma(x)$ for $\sigma = 0.1$ (thinner) and for $\sigma = 0.2$. The shaded areas equal to one.

As the positive parameter σ becomes smaller the graph becomes thinner and taller so that the area under it keeps being equal to one. In the limit $\sigma \to 0^+$ the graph is infinitely thin and tall centered at $x = 0$, see Fig. 26. This prompts the following definition for the δ-function

$$\delta(x-a) = \left\{ \begin{array}{ll} 0\,, & x \neq a \\ \infty\,, & x = a \end{array} \right\} , \qquad \text{and} \quad \int_{-\infty}^{\infty} dx \delta(x) = 1 \,, \qquad a \in \mathbb{R} \,. \tag{E.3}$$

Then

$$\lim_{\sigma \to 0^+} P_\sigma(x) = \delta(x) \,. \tag{E.4}$$

From a purely mathematical viewpoint, the delta-function is not strictly a function, because any extended-real function that is equal to zero everywhere but a single point must have total integral zero. The delta-function only makes sense as a mathematical object when it appears inside integrals. It must be regarded as a kind of limit of a sequence of functions having a tall spike at a point in real line, much like (E.4). Other limits similar to this leading to the δ-function are

$$\begin{aligned} \delta(x) &= \lim_{\epsilon \to 0^+} \frac{\sin(x/\epsilon)}{\pi x} \,, \\ \delta(x) &= \lim_{\epsilon \to 0^+} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} \,, \\ \delta(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \,. \end{aligned} \tag{E.5}$$

One easily establishes the property

$$d(ax) = \frac{\delta(x)}{|a|} \,, \tag{E.6}$$

where a is a real non-zero constant. For a general function $f(x)$ having only N simple zeroes at $x = x_i$, $n = 1, 2, \ldots, N$, this generalizes to

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{\delta(x - x_i)}{|f'(x_i)|} \,. \tag{E.7}$$

For a smooth function $f(x)$ finite at $x = 0$ and for $x \to \infty$ we easily find that

$$\int_{-\infty}^{\infty} dx f(x)\delta(x) = f(0) \,, \qquad \int_{-\infty}^{\infty} dx f(x) \frac{d^n\delta(x)}{dx^n} = (-1)^n \frac{d^n f}{dx^n}\bigg|_{x=0} \,. \tag{E.8}$$

Hence we have the properties

$$f(x)\delta(x) = f(0)\delta(x) \,, \qquad f(x)\frac{d^n\delta(x)}{dx^n} = (-1)^n \frac{d^n f}{dx^n}\bigg|_{x=0} \delta(x) \,. \tag{E.9}$$

E.1 $\delta^{(3)}(x)$ and $\delta^{(2)}(x)$ in curvilinear coordinates

The three dimensional δ-function "centered" at $\mathbf{r}_0 = (x_0, y_0, z_0)$ is defined as

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) = \delta(r - r_0)\delta(y - y_0)\delta(z - z_0) \,. \tag{E.10}$$

Similarly, if polar coordinates coordinates ρ_0, ϕ_0 and z_0 for $\mathbf{r_0}$ are used, the corresponding expression is

$$\delta^{(3)}(\mathbf{r} - \mathbf{r_0}) = \frac{\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(z - z_0)}{\rho} \,. \tag{E.11}$$

Finally, in spherical coordinates r_0, θ_0 and ϕ_0, we have that

$$\delta^{(3)}(\mathbf{r} - \mathbf{r_0}) = \frac{\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)}{r^2 \sin\theta} \,. \tag{E.12}$$

In two dimensions the δ-function "centered" at $\mathbf{r}_0 = (x_0, y_0)$ is defined as

$$\delta^{(2)}(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\delta(y - y_0) \,. \tag{E.13}$$

In polar coordinates this becomes

$$\delta^{(2)}(\mathbf{r}-\mathbf{r_0}) = \frac{\delta(\rho-\rho_0)\delta(\phi-\phi_0)}{\rho} \, . \tag{E.14}$$

Using the above, we may take derivatives of functions more singular than $\frac{1}{r}$ in three dimensional or $\ln\rho$ in two dimensions. The result will necessarily contain derivatives of the delta-function. In manipulating them the property (E.9) is very useful.

E.2 Derivatives leading to delta-functions

A delta function arises when we take derivatives of discontinues functions. To see that we first define the *Heaviside step function*, or just *step function* as

$$\Theta(x) = \begin{cases} 1\,, & x>0 \\ 0\,, & x<0\,. \end{cases} \tag{E.15}$$

This is a discontinuous function and seldom matters what value is used for $\Theta(0)$, since $\Theta(x)$ is usually regarded as a limit of a sequence of functions whose values change very rapidly when we cross from negative to positive values of x. An example is

$$\Theta(x) = \lim_{\epsilon\to 0^+} \Theta_\epsilon(x)\,, \qquad \Theta_e(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{x}{\epsilon}\,. \tag{E.16}$$

In that case $\Theta_\epsilon(0) = \frac{1}{2}$. Graphically, Θ_ϵ is depicted in Fig. 27.

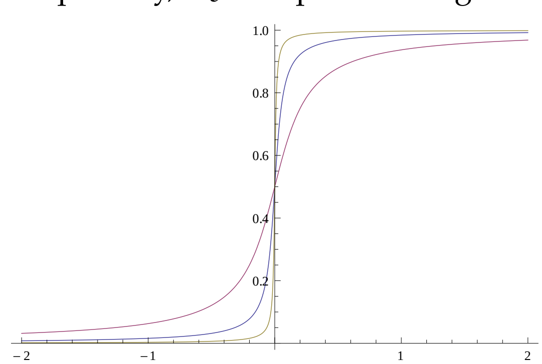

Figure 27: $\Theta_\epsilon(x)$ for $\epsilon = 0.01, 0.05, 0.2$ from up to bottom.

It is easy to show (**exercise**) that

$$\frac{d\Theta(x)}{dx} = \delta(x)\,. \tag{E.17}$$

In particular, note that the derivative $\frac{d\Theta_\epsilon}{dx}$ is given by the function whose limit we take in the second line in (E.5).

In addition, in various computations we have to be careful when we act on functions that become singular at some point in space. This requires a more careful treatment. The typical example is $\nabla^2(1/r)$. A naive computation, without being careful about the behavior at $r = 0$, gives zero (**Exercise**). However, consider the integral over a sphere of radius R centered at $r = 0$

$$\begin{aligned} \int dV \nabla^2 \frac{1}{r} &= \int dV \nabla \cdot \nabla \frac{1}{r} = \oint_S dS \hat{\mathbf{n}} \cdot \nabla \frac{1}{r}\Big|_{r=R} \\ &= R^2 \int d\Omega \frac{-1}{R^2} = -4\pi \ . \end{aligned} \tag{E.18}$$

where I used the divergence theorem, the fact that in our case $\hat{\mathbf{n}} = \hat{r}$ and (D.9). Hence, since everywhere except at $r = 0$, the result of $\nabla^2 \frac{1}{r}$ should be zero, we set

$$\nabla^2 \frac{1}{r} = -4\pi \delta^{(3)}(\mathbf{r}) \ . \tag{E.19}$$

Exercise: Show that

$$\partial_i \partial_j \frac{1}{r} = -\frac{1}{r^3}\left(\delta_{ij} - 3\frac{x_i x_j}{r^2}\right) - \frac{4\pi}{3}\delta_{ij}\delta^{(3)}(\mathbf{r}) \ , \qquad i,j = 1,2,3 \ . \tag{E.20}$$

In two dimensions one encounters a similar problem when taking derivatives of functions that become singular at a point.

Exercise: Show that

$$(\partial_x^2 + \partial_y^2) \ln \rho = 2\pi \delta^{(2)}(\mathbf{r}) \tag{E.21}$$

and then use this result to obtain that

$$\partial_i \partial_j \ln \rho = \frac{1}{\rho^2}\left(\delta_{ij} - 2\frac{x_i x_j}{\rho^2}\right) + \pi \delta_{ij}\delta^{(2)}(\mathbf{r}) \ , \qquad i,j = 1,2 \ . \tag{E.22}$$

F The Legendre eq. and polynomials

The Legendre differential eq. is given by

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right)+\ell(\ell+1)\Theta=0\ . \tag{F.1}$$

Its solutions depend on the interval in which x takes values as well as the values of ℓ. For our purposes it suffices to consider that $-1\leqslant x\leqslant 1$ and that ℓ is a non-negative integer. To get an idea what the solutions arise, consider the case with $\ell=0$. We easily find that two independent solutions of (F.1) are (**exercise**)

$$\Theta=1\ ,\qquad \Theta=\ln\left(\frac{1-x}{1+x}\right)\ . \tag{F.2}$$

The first solution is obviously regular everywhere, but the second one is singular at $x=\pm 1$. This pattern persists for all non-negative integer ℓ's. For all $\ell=0,1,2,\dots$ there is one polynomial solution of order ℓ. A second solution contains the term $\ln\left(\frac{1-x}{1+x}\right)$ and hence it is singular at $x=\pm1$.

As a basis of polynomials of order ℓ we may take the monomials x^m with $m=0,1,\dots,\ell$. However, this basis is not orthogonal in the sense that

$$\int_{-1}^{1}dx x^m x^n\neq 0\ ,\qquad \text{for } m\neq n\ . \tag{F.3}$$

However, one can use the *Gram–Schmidt method* to find an othogonal basis. The result is the Legendre polynomials. Equivalently, they can be defined by the so called *Rodriguez formula*

$$P_\ell(x)=\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\ , \tag{F.4}$$

where they have been normalized so that $P_\ell(1)=1$, $\forall\ \ell=0,1,2,\dots$.

<u>**Exercise:**</u> Show that the $P_\ell(x)$'s as defined above satisfy the Legendre eq. (F.1).

Some of the low order Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1\,, \\
P_1(x) &= x\,, \\
P_2(x) &= \frac{1}{2}(3x^2-1)\,, \\
P_3(x) &= \frac{1}{2}(5x^3-3x)\,, \\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3)\,, \\
P_5(x) &= 1/8(63x^5-70x^3+15x)
\end{aligned} \tag{F.5}$$

and have been drawn in the Fig. 28.

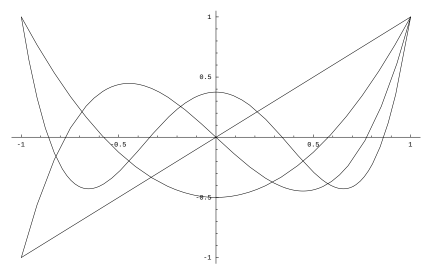

Figure 28: Legendre polynomials $P_\ell(x)$ for $\ell = 1,2,3,4$. Identifying which value of ℓ each curve corresponds to should be apparent.

Remarks:

• Using Rodriguez's formula one finds

$$P_\ell(-x) = (-1)^\ell P_\ell(x)\ . \tag{F.6}$$

• Orthonormality:

$$\int_{-1}^{1} dx\ P_\ell(x)P'_\ell(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'}\ . \tag{F.7}$$

Proof: With no loss of generality assume that $\ell' \leqslant \ell$. Then we use Rodriguez's formula

$$\begin{aligned}
\int_{-1}^{1} dx\ P_\ell(x)P_{\ell'}(x) &= \frac{1}{2^{\ell+\ell'}\ell!\ell'!}\int_{-1}^{1} dx \left[\frac{d^\ell(x^2-1)^\ell}{dx^\ell}\right]\left[\frac{d^{\ell'}(x^2-1)^{\ell'}}{dx^{\ell'}}\right] \\
&= \frac{(-1)^\ell}{2^{\ell+\ell'}\ell!\ell'!}\int_{-1}^{1} dx\ (x^2-1)^\ell \left[\frac{d^{\ell+\ell'}}{dx^{\ell+\ell'}}(x^2-1)^{\ell'}\right]\ ,
\end{aligned}$$

where I performed ℓ integrations by parts. Because $\ell+\ell'$ derivatives act on a polynomials of order $2\ell'$ for a non-vanishing result we should have that $\ell \leqslant \ell'$. Hence $\ell=\ell'$. In that case we have that

$$\begin{aligned}\int_{-1}^{1} dx\, P_\ell^2(x) &= \frac{(-1)^\ell}{2^{2\ell}(\ell!)^2}\int_{-1}^{1} dx\,(x^2-1)^\ell \frac{d^{2\ell}(x^2-1)^\ell}{dx^{2\ell}} \\ &= \frac{(-1)^\ell(2\ell)!}{2^{2\ell}(\ell!)^2}\int_{-1}^{1} dx\,(x^2-1)^\ell\,.\end{aligned}$$

Computing the last integral we find that (**exercise**)

$$\int_{-1}^{1} dx\, P_\ell^2(x) = \frac{2}{2\ell+1}\,.$$

• The set of $P_\ell(x)$ for $\ell=0,1,2,\dots$ form a complete set in $x\in[-1,1]$. This means that any function $f(x)$ with $x\in[-1,1]$ can be expanded as

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)\,, \tag{F.8}$$

with

$$A_\ell = \left(\ell+\frac{1}{2}\right)\int_{-1}^{1} dx\, f(x)P_\ell(x)\,. \tag{F.9}$$

One may easily verify that this implied and implies that

$$\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(x)P_\ell(x') = \delta(x-x')\,. \tag{F.10}$$

• The generating function

$$\frac{1}{\sqrt{t^2+1-2tx}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)\,, \qquad t<1\,. \tag{F.11}$$

A proof of that with a physical interpretation is given in the last part of section 3.

• One may prove the recursive relations

$$\begin{aligned}(2\ell+1)P_\ell(x) &= P'_{\ell+1}(x) - P'_{\ell-1}(x)\,,\\ (2\ell+1)xP_\ell(x) &= (\ell+1)P_{\ell+1}(x)+\ell P_{\ell-1}(x)\,.\end{aligned} \tag{F.12}$$

Proof: First we take the derivative of the generating function w.r.t. t

$$\frac{x-t}{(t^2+1-2tx)^{3/2}} = \sum_{\ell=0}^{\infty} \ell P_\ell(x) t^{\ell-1} \ . \tag{F.13}$$

Then we multiply with $2t$ and add the generating function

$$\frac{1-t^2}{(t^2+1-2tx)^{3/2}} = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x) t^{\ell} \ .$$

From the generating function taking the derivative w.r.t. x

$$\frac{t}{(t^2+1-2tx)^{3/2}} = \sum_{\ell=0}^{\infty} P'_\ell(x) t^{\ell} \ . \tag{F.14}$$

Dividing and multiplying with t and then subtracting the two results we find that

$$\frac{1-t^2}{(t^2+1-2tx)^{3/2}} = \sum_{\ell=0}^{\infty} t^{\ell} [P'_{\ell+1}(x) - P'_{\ell-1}(x)] \ , \tag{F.15}$$

after some index rearrangements. Equating we find the first of the recursive relations in (F.12). The proof of the second recursive relation is left as an (**exercise**).

Suggested bibliography

[1] D.J. Griffiths, *Introduction to Electrodynamics* (standard textbook).

[2] J.D. Jackson, *Classical Electrodynamics* (more advanced).